Invariant Differential Operators and Characters of the AdS$_4$ Algebra

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Abstract

The aim of this paper is to apply systematically to AdS$_4$ some modern tools in the representation theory of Lie algebras which are easily generalised to the supersymmetric and quantum group settings and necessary for applications to string theory and integrable models. Here we introduce the necessary representations of the AdS$_4$ algebra and group. We give explicitly all singular (null) vectors of the reducible AdS$_4$ Verma modules. These are used to obtain the AdS$_4$ invariant differential operators. Using this we display a new structure - a diagram involving four partially equivalent reducible representations one of which contains all finite-dimensional irreps of the AdS$_4$ algebra. We study in more detail the cases involving UIRs, in particular, the Di and the Rac singletons, and the massless UIRs. In the massless case we discover the structure of sets of $2s_0 - 1$ conserved currents for each spin s_0 UIR, $s_0 = 1, \frac{3}{2}, \ldots$ All massless cases are contained in a one-parameter subfamily of the quartet diagrams mentioned above, the parameter being the spin s_0. Further we give the classification of the $so(5,\mathbb{C})$ irreps presented in a diagrammatic way which makes easy the derivation of all character formulae. The paper concludes with a speculation on the possible applications of the character formulae to integrable models.

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1. Introduction

The AdS/CFT correspondence between gravitational and gauge forces in general states that a bulk supergravity theory resp. closed string theory is equivalent to a Yang-Mills resp. open string theory on the boundary of space-time (holography). This correspondence

is best understood for four-dimensional N=4 supersymmetric Yang-Mills, which is dual to supergravity on AdS5, and which can be explained by D3-branes in type IIB superstring theory. It is also important that string theories have conformal symmetry and thus may be viewed as ultraviolet limits of integrable (but not conformally invariant) models; or the latter may be viewed as deformations, including quantum group deformations, of conformal models. However, much work remains to be done in order to establish analogous results for dimensions $D \neq 4$ and also for deformations of the original anti de Sitter backgrounds following recent developments in [1--7]. In particular, for $D = 3$ we need to develop the representation theory of the AdS_4 algebra $so(3,2)$ in a formalism that will be suitable for the purposes of the AdS/CFT correspondence, supersymmetrizations and q-deformations along the lines of [8--19].

The AdS$_4$ algebra $so(3,2)$ attracted attention very early since it has truly remarkable unitary representations known as singletons, which were first discovered by Dirac in 1963 [20]. These representations have been extensively studied by Fronsdal, Flato and Evans [21--28]. There are two singleton representations, called Di and Rac. In terms of the lowest energy value E_0 and the spin s_0, the Di has $E_0 = 1,\ s_0 = 1/2$ while the Rac has $E_0 = 1/2,\ s_0 = 0$. These representations have remarkably reduced spectrum (weight spaces). Consequently, the singleton field theory has a very large gauge symmetry which enables one to gauge away the singleton fields everywhere except on the boundary of the anti de Sitter space [27]. Moreover, the direct product of two singletons decomposes into infinitely many massless states of the anti de Sitter group [25].

However, the above results and the methods applied are not suitable for the envisaged generalizations. Thus, the aim of this paper is to apply systematically some modern tools in the representation theory of Lie algebras [29],[30],[31] which are easily generalised to the supersymmetric and quantum group settings [32]. These generalizations will be done in sequel(s) of the present paper taking into account also work done for the AdS$_4$ algebra in [33], its superpartners [34], its Kac-Moody counterpart [35], and its quantum group deformation [36]. For $so(d,2)$, $d > 4$, similar technique was applied in [37].

The paper is organized as follows. In Section 2 we give preliminaries including facts about the $so(3,2)$ algebra and the group $SO_0(3,2)$. In Section 3 we adapt the approach of [29] to the current setting introducing the elementary representations (ERs) of $SO_0(3,2)$, relating them to the Verma modules over the complexification $so(5,\mathbb{C})$ of $so(3,2)$. Then we give the singular (null) vectors of the reducible Verma modules and use them (a la [29]) to obtain the invariant differential operators between the reducible ERs. In Section 4 we apply these differential operators to the cases involving UIRs. For the massless case we discover the structure of sets of $2s_0 - 1$ conserved currents for each spin s_0 UIR, $s_0 = 1, \frac{3}{2}, \ldots$ Another structure parametrized by the same values of s_0 is a quartet diagram of partially equivalent reducible ERs involving the massless case for that s_0, a massive UIR (of spin s_0-1) and an ER containing finite-dimensional irrep of dimension $s_0(4s_0^2-1)/3$. In Section 5 we give the classification of the $so(5,\mathbb{C})$ irreps. This is presented in a diagrammatic way which makes easy the derivation of the character formulae of these irreps which is done in Sec. 6. Section 6 is concluded with a speculation on the possible applications of the derived character formulae to integrable models.

2. Preliminaries

2.1. Lie algebra. We start with the complexification $\mathcal{G}^{\mathbb{C}} = so(5,\mathbb{C}) = B_2$ of the algebra $\mathcal{G} = so(3,2)$. We use the standard definition of $\mathcal{G}^{\mathbb{C}}$ given in terms of the Chevalley generators $X_i^\pm$, H_i , $i = 1, 2$, by the relations :

$$[H_j\ ,\ H_k] \ =\ 0\ ,\quad [H_j\ ,\ X_k^\pm] \ =\ \pm a_{jk} X_k^\pm\ ,\quad [X_j^+\ ,\ X_k^-] \ =\ \delta_{jk}\ H_j\ , \tag{2.1a}$$

$$\sum_{m=0}^{n} (-1)^m \binom{n}{m} \left(X_j^\pm\right)^m X_k^\pm \left(X_j^\pm\right)^{n-m} \ =\ 0\ ,\quad j \neq k\ ,\quad n = 1 - a_{jk}\ , \tag{2.1b}$$

where

$$(a_{jk}) \ =\ (\alpha_j^\vee, \alpha_k) \ =\ \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \tag{2.2}$$

is the Cartan matrix of $\mathcal{G}^{\mathbb{C}}$, $\alpha_j^\vee \equiv \frac{2\alpha_j}{(\alpha_j,\alpha_j)}$ is the co-root of α_j , $(\cdot,\cdot)$ is the scalar product of the roots, so that the non-zero products between the simple roots are: $(\alpha_1,\alpha_1) = 2$, $(\alpha_2,\alpha_2) = 4$, $(\alpha_1,\alpha_2) = -2$. The elements H_i span the Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}^{\mathbb{C}}$, while the elements $X_i^\pm$ generate the subalgebras $\mathcal{G}^\pm$. We shall use the standard triangular decomposition

$$\mathcal{G}^{\mathbb{C}} \ =\ \mathcal{G}_+ \otimes \mathcal{H} \oplus \mathcal{G}_-\ ,\quad \mathcal{G}_\pm \ \equiv \ \mathop{\oplus}_{\alpha\in\Delta^\pm} \mathcal{G}_\alpha\ , \tag{2.3}$$

where Δ^+, Δ^-, are the sets of positive, negative, roots, resp., and $\dim\, \mathcal{G}_\alpha = 1$. Explicitly we have:

$$\Delta^\pm = \{\pm\alpha_1\ ,\ \pm\alpha_2\ ,\ \pm\alpha_3\ ,\ \pm\alpha_4\}\ ,\qquad \alpha_3 = \alpha_1 + \alpha_2\ ,\quad \alpha_4 = 2\alpha_1 + \alpha_2\ . \tag{2.4}$$

Let us denote the root space vector of $\mathcal{G}_\alpha$ by X_α, or more explicitly: $X_k^\pm \equiv X_{\pm\alpha_k}$, $k = 1, 2, 3, 4$. To give the full Cartan-Weyl basis we need to define also $X_k^\pm$, $k = 3, 4$, for which we follow [38]:

$$X_3^\pm = \pm[X_1^\pm, X_2^\pm],\qquad X_4^\pm = \pm\tfrac{1}{2}\,[X_1^\pm, X_3^\pm]\ . \tag{2.5}$$

Then we have:

$$H_3 \ \equiv \ [X_3^+, X_3^-] \ =\ H_1 + 2H_2\ ,\quad H_4 \ \equiv \ [X_4^+, X_4^-] \ =\ H_1 + H_2\ . \tag{2.6}$$

Note that for H_k also holds:

$$\lambda(H_k) \ =\ (\lambda, \alpha_k^\vee)\ ,\qquad \forall\, \lambda \in \mathcal{H}^*\ ,\quad k = 1, 2, 3, 4. \tag{2.7}$$

The algebra $\mathcal{G} = so(3,2)$ is a maximally split real form [39] of $\mathcal{G}^{\mathbb{C}}$ so we can use the same basis (but over $\mathbb{R}$) and the same root system. Thus, we can use the second order Casimir of $\mathcal{G}^{\mathbb{C}}$:

$$\begin{aligned}\mathcal{C}_2 \ &= \tfrac{1}{2}(X_1^+X_1^- + X_1^-X_1^+) \ +\ X_2^+X_2^- + X_2^-X_2^+ \ +\ \tfrac{1}{2}(X_3^+X_3^- + X_3^-X_3^+) \ + \\ &+\ X_4^+X_4^- + X_4^-X_4^+ \ +\ \tfrac{1}{2}H_1^2 \ +\ H_2^2 \ +\ H_1H_2\ .\end{aligned} \tag{2.8}$$

It useful to relate the Cartan-Weyl basis given above with the $so(3,2)$ generators X_{AB} :

$$\begin{aligned}
H_1 &= 2iX_{12} \ , \quad X_1^{\pm} \ = \ X_{10} \pm iX_{20} \ , && (2.9a)\\
H_2 &= X_{34} - iX_{12} \ , \quad X_2^{\pm} \ = \ \mp\tfrac{1}{2}\left(X_{23} \pm X_{24} + iX_{14} \pm iX_{13}\right) \ , && (2.9b)\\
X_3^{\pm} &= \mp i\left(X_{03} \pm X_{04}\right) \ , && (2.9c)\\
X_4^{\pm} &= \tfrac{1}{2}\left(X_{24} \pm X_{23} \mp iX_{14} - iX_{13}\right) \ . && (2.9d)
\end{aligned}$$

It is easy to see that the ten generators $X_{AB} = -X_{BA}$, $A, B = 0,1,2,3,4$, satisfy the standard $so(3,2)$ commutation relations:[1]

$$[X_{AB}, X_{CD}] \ = \ \eta_{AC}X_{BD} + \eta_{BD}X_{AC} - \eta_{AD}X_{BC} - \eta_{BC}X_{AD} \ . \qquad (2.10)$$

where $\eta_{11} = \eta_{22} = \eta_{33} = -\eta_{00} = -\eta_{44} = 1$, $\eta_{jk} = 0$ if $j \neq k$.

2.2. Finite-dimensional realization. It is useful to have a finite-dimensional realization of $\mathcal{G}$ which we take from [30] (cf. (2.18) which we restrict from $so(4,2)$ to $so(3,2)$):[2]

$$\begin{aligned}
X_{12} &= -\tfrac{i}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} , \qquad X_{a3} \ = \ \tfrac{i}{2}\begin{pmatrix} 0 & \sigma_a \\ \sigma_a & 0 \end{pmatrix} , \ a = 1,2, \\
X_{04} &= \tfrac{i}{2}\begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} ; && (2.11a)\\
X_{34} &= \tfrac{1}{2}\begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} , \qquad X_{a0} \ = \ \tfrac{1}{2}\begin{pmatrix} \sigma_a & 0 \\ 0 & -\sigma_a \end{pmatrix} , \ a = 1,2, && (2.11b)\\
X_{03} &= \tfrac{i}{2}\begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} , \qquad X_{a4} \ = \ \tfrac{i}{2}\begin{pmatrix} 0 & \sigma_a \\ -\sigma_a & 0 \end{pmatrix} , \ a = 1,2.
\end{aligned}$$

The hermiticity properties of this defining realisation are:

$$\begin{aligned}
X_{AB}^{\dagger} &= -X_{AB} \quad \text{for} \quad (A,B) \ = \ (0,4), (j,k), && (2.12a)\\
X_{AB}^{\dagger} &= X_{AB} \quad \text{for} \quad (A,B) \ = \ (0,j), (j,4), && (2.12b)
\end{aligned}$$

where $j,k = 1,2,3$. Note that the four generators in (2.12a) (or (2.11a)) are compact. They span the maximal compact subalgebra $\mathcal{K} = so(3) \oplus so(2)$, the $so(3)$ being spanned by the generators X_{jk} , $j,k = 1,2,3$, the $so(2)$ being spanned by the generator X_{04} . The six generators in (2.12b) (or (2.11b)) are non-compact.

Thus, in this basis, we can identify - up to sign - the Hermitian conjugation with the Cartan involution θ which in general is defined by:

$$\begin{aligned}
\theta : X \mapsto X \ , \quad &\text{if} \quad X \quad \text{is compact} \ , \\
\theta : X \mapsto -X, \quad &\text{if} \quad X \quad \text{is noncompact} \ .
\end{aligned} \qquad (2.13)$$

[1] Note that often are used the generators $M_{AB} = iX_{AB}$.

[2] It is different from the one in [34] which we used in [33].

From the above we have explicitly a finite-dimensional representation for the Cartan-Weyl basis:

$$H_1 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} , \quad X_1^+ = \begin{pmatrix} \sigma_+ & 0 \\ 0 & -\sigma_+ \end{pmatrix} , \quad X_1^- = \begin{pmatrix} \sigma_- & 0 \\ 0 & -\sigma_- \end{pmatrix} , \tag{2.14a}$$

$$H_2 = \begin{pmatrix} e_2 & 0 \\ 0 & -e_1 \end{pmatrix} , \quad X_2^+ = \begin{pmatrix} 0 & \sigma_- \\ 0 & 0 \end{pmatrix} , \quad X_2^- = \begin{pmatrix} 0 & 0 \\ \sigma_+ & 0 \end{pmatrix} , \tag{2.14b}$$

$$X_3^+ = \begin{pmatrix} 0 & 1_2 \\ 0 & 0 \end{pmatrix} , \quad X_3^- = \begin{pmatrix} 0 & 0 \\ 1_2 & 0 \end{pmatrix} , \quad H_3 = H_1 + 2H_2 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} , \tag{2.14c}$$

$$X_4^+ = \begin{pmatrix} 0 & \sigma_+ \\ 0 & 0 \end{pmatrix} , \quad X_4^- = \begin{pmatrix} 0 & 0 \\ \sigma_- & 0 \end{pmatrix} , \quad H_4 = H_1 + H_2 = \begin{pmatrix} e_1 & 0 \\ 0 & -e_2 \end{pmatrix} , \tag{2.14d}$$

$$e_1 \equiv \tfrac{1}{2}(1+\sigma_3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} , \quad e_2 \equiv \tfrac{1}{2}(1-\sigma_3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} ,$$

$$\sigma_+ \equiv \tfrac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad \sigma_- \equiv \tfrac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .$$

2.3. Structure theory. We need some more structure theory related to the applications to conformal field theory, namely, we need the following (Bruhat) decomposition

$$\mathcal{G} \ = \ \mathcal{N}_+ \oplus \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}_- \tag{2.15}$$

in which all four subalgebras have physical meaning, namely, the subalgebra $\mathcal{M}$ is the Lorentz subalgebra of three-dimensional Minkowski space-time M^3, i.e., $\mathcal{M} = so(2,1)$, the subalgebras $\mathcal{N}_+, \mathcal{N}_-$ are abelian and represent the translations of M^3 and special conformal transformations of M^3, and the algebra $\mathcal{A}$ represents the dilatations of M^3 ($\mathcal{A}$ commutes with $\mathcal{M}$). Explicitly, we have:

$$\mathcal{M} \ : \ \{X_{12}\,,\ X_{a3}\,,\ a=1,2\}\,, \quad \mathcal{A} \ : \ \{H_3\}\,, \quad \mathcal{N}_\pm \ : \ \{X_k^\pm\,, \quad k=2,3,4\}\ . \tag{2.16}$$

Note that the generators $H_1\,,\ X_1^\pm$ are complex linear combinations of those of $\mathcal{M}$ (cf. (2.9a)), i.e., span the complexification $\mathcal{M}^{\mathbb{C}} = so(3,\mathbb{C})$ of $\mathcal{M}$, and actually represent a triangular decomposition of $\mathcal{M}^{\mathbb{C}} \ = \ \mathcal{M}^{\mathbb{C}}_+ \oplus \mathcal{M}_h \oplus \mathcal{M}^{\mathbb{C}}_-$: $X_1^\pm$ spanning $\mathcal{M}^{\mathbb{C}}_\pm$, H_1 spanning the Cartan subalgebra $\mathcal{M}_h$. Matters are arranged so that the factors from (2.3) are related to the complexification of (2.15) in the following obvious manner:

$$\mathcal{G}_\pm \ = \ \mathcal{N}_\pm \oplus \mathcal{M}^{\mathbb{C}}_\pm\ , \quad \mathcal{H} \ = \ \mathcal{A} \oplus \mathcal{M}_h\ . \tag{2.17}$$

2.4. Lie groups. Finally, we introduce the corresponding connected Lie groups: $G = SO_0(3,2)$ with Lie algebra $\mathcal{G} = so(3,2)$, $K = SO(3)\times SO(2)$ is the maximal compact subgroup of G, $A = \exp(\mathcal{A}) = SO_0(1,1)$ is abelian simply connected, $N_\pm = \exp(\mathcal{N}_\pm)$ are abelian simply connected subgroups of G preserved by the action of A. The group $M \cong SO_0(2,1)$ (with Lie algebra $\mathcal{M}$) commutes with A. The subgroup $P = MAN$, where $N = N_+$ or $N = N_-$, is a *maximal parabolic subgroup* of G. Parabolic subgroups are important because the representations induced from them generate all admissible irreducible representations of G [40],[41].

3. Representations and invariant operators

3.1. Elementary representations. We use the approach of [29] which we adapt in a condensed form here. We work with so-called *elementary representations* (ERs). They are induced from representations of $P = MAN_+$, with the factor N_+ being represented trivially. We take $p = 0, 1, 2, \dots$ to fix a $(p+1)$-dimensional representation of M, and $c \in \mathbb{C}$ to fix a (non-unitary) character of A. This data is enough to determine a weight $\Lambda \in \mathcal{H}^*$ as follows: $\Lambda(H_1) = p$, $\Lambda(H_3) = c$. Thus, we shall denote the ERs by C^Λ. They can be considered also *holomorphic* elementary representations of $G^{\mathbb{C}}$ and their functions can be taken to be complex-valued C^∞ functions on G or $G^{\mathbb{C}}$. The representation action is given by the *left regular action*, which infinitesimally is:

$$(\pi_\Lambda(X)\,\varphi)(g) \;\doteq\; \frac{d}{dt}\varphi(\exp(-tX)g)|_{t=0} \tag{3.1}$$

where $X \in \mathcal{G}$, $g \in G$, and these can be extended to $\mathcal{G}^{\mathbb{C}}$ and $G^{\mathbb{C}}$. These functions possess the properties of right covariance [29] which here means:

$$\hat{X}\varphi \;=\; \Lambda(X)\cdot\varphi\ , \quad X \in \mathcal{H} \tag{3.2a}$$

$$\hat{X}\varphi \;=\; 0\ , \quad X \in \mathcal{G}_+ \tag{3.2b}$$

where $\hat{X}$ is the *right* action of the generators of the algebra $\mathcal{G}$

$$(\hat{X}\varphi)(g) \;\doteq\; \frac{d}{dt}\varphi(g\exp(tX))|_{t=0} \tag{3.3}$$

Right covariance is used also to pass from functions on the group G to the so-called *reduced* functions $\hat{\varphi}$ on the coset space $G^{\mathbb{C}}/B$, where $B = \exp(\mathcal{H})\exp(\mathcal{G}_+)$ is a Borel subgroup of $G^{\mathbb{C}}$, (and we use the fact that the group G is maximally split and we can work with $G^{\mathbb{C}}$ instead). Note that $G^{\mathbb{C}}/B$ is a completion of $G_- = \exp(\mathcal{G}_-)$ and as usually we shall use the local coordinates of G_- :

$$G_- \;=\; \left\{ g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ z & 1 & 0 & 0 \\ \xi + zv/2 & v & 1 & 0 \\ \eta - vz^2/6 & \xi - zv/2 & -z & 1 \end{pmatrix} \right\} \tag{3.4}$$

obtained from exponentiation of the general term: $zX_1^- + vX_2^- + \xi X_3^- + \eta X_4^-$ of $\mathcal{G}_-$. The functions $\hat{\varphi}(z, v, \xi, \eta)$ are polynomials in the variable z of degree p and smooth functions in the other three variables. Consistently with the above, we have:

$$\hat{H}_1\,\hat{\varphi} \;=\; p\,\hat{\varphi}\ , \quad p = 0, 1, \dots\ , \tag{3.5a}$$

$$\hat{H}_3\,\hat{\varphi} \;=\; c\,\hat{\varphi}\ . \tag{3.5b}$$

The right action of $\mathcal{G}_-$ on $\hat{\varphi}$ is calculated easily and after some changes of variables $(z, v, \xi, \eta) \mapsto (z, v, x, u)$ we obtain:

$$\hat{X}_1^- \;=\; \partial_z\ , \quad \hat{X}_2^- \;=\; \partial_v - z\partial_x + z^2\partial_u\ , \quad \hat{X}_3^- \;=\; \partial_x - 2z\partial_u\ , \quad \hat{X}_4^- \;=\; \partial_u\ . \tag{3.6}$$

The left, representation, action on $\hat{\varphi}$ is derived in a straightforward way and after the same changes of variables we have:

$$\begin{aligned}
\pi_\Lambda(X_1^+) &= z^2\partial_z + 2x\partial_v + u\partial_x - zp && (3.7a)\\
\pi_\Lambda(X_2^+) &= -(x+zv)\partial_z + v^2\partial_v + vx\partial_x + x^2\partial_u - \tfrac{1}{2}(c-p)v && (3.7b)\\
\pi_\Lambda(H_1) &= 2z\partial_z - 2v\partial_v + 2u\partial_u - p && (3.7c)\\
\pi_\Lambda(H_2) &= -z\partial_z + 2v\partial_v + x\partial_x - \tfrac{1}{2}(c-p) && (3.7d)\\
\pi_\Lambda(X_1^-) &= -\partial_z + v\partial_x + 2x\partial_u && (3.7e)\\
\pi_\Lambda(X_2^-) &= -\partial_v && (3.7f)
\end{aligned}$$

In addition, we have:

$$\begin{aligned}
\pi_\Lambda(X_3^+) &= [\pi_\Lambda(X_1^+), \pi_\Lambda(X_2^+)] = (vz^2-u)\partial_z + 2xv\partial_v + (x^2+uv)\partial_x + \\
&\quad + 2xu\partial_u - (x+zv)p - (c-p)x && (3.7g)\\
\pi_\Lambda(X_3^-) &= [\pi_\Lambda(X_2^-), \pi_\Lambda(X_1^-)] = -\partial_x && (3.7h)\\
\pi_\Lambda(H_3) &= [\pi_\Lambda(X_3^+), \pi_\Lambda(X_3^-)] = \pi_\Lambda(H_1) + 2\pi_\Lambda(H_2) = \\
&= 2x\partial_x + 2v\partial_v + 2u\partial_u - c && (3.7i)\\
\pi_\Lambda(X_4^+) &= \tfrac{1}{2}[\pi_\Lambda(X_1^+), \pi_\Lambda(X_3^+)] = (xz^2+uz)\partial_z + x^2\partial_v + xu\partial_x + \\
&\quad + u^2\partial_u - (xz+u)p - \tfrac{1}{2}(c-p)u && (3.7j)\\
\pi_\Lambda(X_4^-) &= [\pi_\Lambda(X_3^-), \pi_\Lambda(X_1^-)] = -\partial_u && (3.7k)\\
\pi_\Lambda(H_4) &= [\pi_\Lambda(X_4^+), \pi_\Lambda(X_4^-)] = \pi_\Lambda(H_1) + \pi_\Lambda(H_2) = \\
&= z\partial_z + x\partial_x + 2u\partial_u - \tfrac{1}{2}(c+p) && (3.7l)
\end{aligned}$$

Since the left and right actions commute we can calculate the value of the second order Casimir in either of them and obtain:

$$\pi_\Lambda(\mathcal{C}_2) = \hat{\mathcal{C}}_2 = \tfrac{1}{4}\left((p+1)^2 + (c+3)^2\right) . \qquad (3.8)$$

Remark: Note that starting from a constant function $\hat{\varphi} = C$ and taking $c = p$ we obtain finite-dimensional representation consisting of polynomials in z obtained by the action of $\pi_\Lambda(X_1^+)$:

$$(\pi_\Lambda(X_1^+))^k \cdot C = \begin{cases} (-z)^k \frac{p\,!}{(p-k)!}\, C & k \leq p \\ 0 & k > p \end{cases} \qquad (3.9)$$

The same would be true if $p = 0$ and $\frac{1}{2}(c-p)$ would be a non-negative integer r, then we would obtain finite-dimensional representation consisting of polynomials in v obtained by the action of $\pi_\Lambda(X_2^+)$:

$$(\pi_\Lambda(X_2^+))^k \cdot C = \begin{cases} (-v)^k \frac{r\,!}{(r-k)!}\, C & k \leq r \\ 0 & k > r \end{cases} \qquad (3.10)$$

3.2. Verma modules and singular vectors. We note that conditions (3.2) are the defining conditions for the highest weight vector of a highest weight module (HWM) over $\mathcal{G}^{\mathbb{C}}$ with highest weight Λ, in particular, of a Verma module with this highest weight. Let us recall that a *Verma module* V^Λ is defined as the HWM over $\mathcal{G}^{\mathbb{C}}$ with highest weight $\Lambda \in \mathcal{H}^*$ and highest weight vector $v_0 \in V^\Lambda$, induced from the one-dimensional representation $V_0 \cong \mathbb{C} v_0$ of $U(\mathcal{B})$, where $\mathcal{B} = \mathcal{H} \oplus \mathcal{G}_+$ is a Borel subalgebra of $\mathcal{G}$, such that:

$$\begin{aligned} X\ v_0 &= 0, \quad \forall\, X \in \mathcal{G}_+ \\ H\ v_0 &= \Lambda(H)\ v_0\,, \quad \forall\, H \in \mathcal{H} \end{aligned} \tag{3.11}$$

Verma modules are generically irreducible. A Verma module V^Λ is reducible [42] iff there exists a root $\beta \in \Delta^+$ and $m \in \mathbb{N}$ such that

$$(\Lambda + \rho\ ,\ \beta^\vee)\ =\ m \tag{3.12}$$

holds, where $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha$. If (3.12) holds then V^Λ is reducible. It contains an invariant submodule which is also a Verma module $V^{\Lambda'}$ with shifted weight $\Lambda' = \Lambda - m\beta$. This statement is equivalent to the fact that V^Λ contains a *singular vector* $v_s \in V^\Lambda$, such that $v_s \neq \xi v_0$, $(0 \neq \xi \in \mathbb{C})$, and :

$$\begin{aligned} X\ v_s &= 0, \quad \forall\, X \in \mathcal{G}_+ \\ H\ v_s &= \Lambda'(H)\ v_s\,, \quad \Lambda' \ =\ \Lambda - m\beta, \quad \forall\, H \in \mathcal{H} \end{aligned} \tag{3.13}$$

It is important that one can find explicit formulae for the singular vectors. The singular vectors introduced above are given by [29] :

$$v_s \ =\ v^{\alpha,m_\alpha} \ =\ \mathcal{P}^{\alpha,m_\alpha}(\mathcal{G}_-)\ v_0 \tag{3.14}$$

where $\mathcal{P}^{\alpha,m_\alpha}$ is a homogeneous polynomial in the generators of $\mathcal{G}_-$ of weight $\alpha\, m_\alpha$ and is unique up to a non-zero multiplicative constant. The conditions (3.12) spelled out for the four positive roots in our situation are:

$$\begin{aligned} m_1 &= m_1(\Lambda) \doteq \Lambda(H_1) + 1 \ \in \mathbb{N}\ , && (3.15a) \\ m_2 &= m_2(\Lambda) \doteq \Lambda(H_2) + 1 \ \in \mathbb{N}\ , && (3.15b) \\ m_3 &= m_3(\Lambda) \doteq \Lambda(H_3) + 3 = m_1 + 2m_2 \ \in \mathbb{N}\ , && (3.15c) \\ m_4 &= m_4(\Lambda) \doteq \Lambda(H_4) + 2 = m_1 + m_2 \ \in \mathbb{N}\ . && (3.15d) \end{aligned}$$

The singular vectors corresponding to these cases are :

$$\begin{aligned} v^{\alpha_1,m_1} &= (X_1^-)^{m_1}\, v_0\ , \quad m_1 \in \mathbb{N}\ , && (3.16a) \\ v^{\alpha_2,m_2} &= (X_2^-)^{m_2}\, v_0\ , \quad m_2 \in \mathbb{N}\ , && (3.16b) \\ v^{\alpha_3,m_3} &= \sum_{k=0}^{m_3} a_k\ (X_2^-)^{m_3-k}\,(X_1^-)^{m_3}\,(X_2^-)^k\, v_0\ , \quad m_3 \in \mathbb{N}\ , && (3.16c) \end{aligned}$$

$$a_k \;=\; \begin{cases} a_0\,(-1)^k \binom{m_3}{k} \frac{m_2}{m_2-k} \ , & m_2 \notin \{\,0,\ldots,m_3\,\} \\ a_0\ \delta_{k,m_2} \ , & m_2 \in \{\,0,\ldots,m_3\,\} \end{cases} \tag{3.16c'}$$

$$v^{\alpha_4,m_4} \;=\; \sum_{k=0}^{2m_4} b_k\ (X_1^-)^{2m_4-k}\,(X_2^-)^{m_4}\,(X_1^-)^k\,v_0 \ , \quad m_4 \in I\!\!N \ , \tag{3.16d}$$

$$b_k \;=\; \begin{cases} b_0\,(-1)^k \binom{2m_4}{k} \frac{m_1}{m_1-k} \ , & m_1 \notin \{\,0,\ldots,2m_4\,\} \\ b_0\ \delta_{k,m_1} \ , & m_1 \in \{\,0,\ldots,2m_4\,\} \end{cases} \tag{3.16d'}$$

(Note that in each of the four cases (3.16) only the relevant m_j must be a natural number (as displayed).) Formulae ($3.16a,b$) are the general expressions valid for any simple root [29], while ($3.16c,d$) are given in [38]. Certainly, (3.12) may be fulfilled for several positive roots, even for all of them if (3.12) is fulfilled for all simple roots.

For future use we note one important special case of ($3.16c$) when m_3 is even, and $m_1 = 1$, then $m_2 = \frac{1}{2}(m_3-1) \notin Z\!\!\!Z$. In this case ($3.16c$) can be written as follows:

$$v^{\alpha_3,m_3} \;=\; \left((X_3^-)^2 - 4X_4^-X_2^-\right)^{\frac{1}{2}m_3}\ v_0 \;+\; P(\mathcal{G}_-)\ X_1^-\ v_0 \ , \qquad m_3 \in 2I\!\!N,\ m_1 = 1 \ , \tag{3.17}$$

where in order to achieve this simple form we have introduced also the non-simple root vectors X_3^-, X_4^-, and $P(\mathcal{G}_-)$ is a polynomial whose exact form is not important for our purposes.

Note that independently from the question of reducibility the numbers m_1, m_2 may be used to characterize the representations C^Λ and V^Λ. In particular, the second order Casimir is expressed in a simple manner through these numbers after evaluation on the highest weight vector:

$$\begin{aligned} \mathcal{C}_2\,v_0 \;&= \{\, \tfrac{1}{2}(X_1^+X_1^- + X_1^-X_1^+) \;+\; X_2^+X_2^- + X_2^-X_2^+ \;+\; \tfrac{1}{2}(X_3^+X_3^- + X_3^-X_3^+) \;+ \\ &\quad +\; X_4^+X_4^- + X_4^-X_4^+ \;+\; \tfrac{1}{2}H_1^2 \;+\; H_2^2 \;+\; H_1H_2\,\}\,v_0 \;= \\ &= \{\, \tfrac{1}{2}H_1 \;+\; H_2 \;+\; \tfrac{1}{2}H_3 \;+ H_4 \;+\; \tfrac{1}{2}H_1^2 \;+\; H_2^2 \;+\; H_1H_2\,\}\,v_0 \;= \\ &= \left(\, \tfrac{1}{2}m_1^2 + m_2^2 + m_1m_2 \,\right)\,v_0 \;= && (3.18a) \\ &= \tfrac{1}{4}\left(\, m_1^2 + m_3^2 \,\right)\,v_0 \;= && (3.18b) \\ &= \tfrac{1}{2}\left(\, m_2^2 + m_4^2 \,\right)\,v_0 \ . && (3.18c) \end{aligned}$$

(Passing from the first to the second equality we use the commutation relations and the defining properties of v_0, e.g., $(X_1^+X_1^-{+}X_1^-X_1^+)\,v_0 \;=\; X_1^+X_1^-\,v_0 \;=\; H_1\,v_0$.) Comparison with (3.8) is straightforward if we substitute in ($3.18b$) $m_1 = \Lambda(H_1)+1 = p+1$, $m_3 = \Lambda(H_3)+3 = c+3$.

It is useful for future reference to write down the numbers corresponding to the four cases of Λ', i.e., to write down $m_1(\Lambda'), m_2(\Lambda')$ for the four cases of (3.15). In fact, though it is redundant, the picture becomes more instructive, if we write down also $m_3(\Lambda'), m_4(\Lambda')$. Explicitly, we have for $m_i \equiv m_i(\Lambda)$:

$$(m_1(\Lambda'), m_2(\Lambda'), m_3(\Lambda'), m_4(\Lambda')) \;=\; (-m_1, m_4, m_3, m_2) \ ,$$

$$
\begin{aligned}
&& \Lambda' = \Lambda - m_1\alpha_1, \quad m_1 \in I\!N \ , \qquad (3.19a)\\
(m_1(\Lambda'), m_2(\Lambda'), m_3(\Lambda'), m_4(\Lambda')) &= (m_3, -m_2, m_1, m_4) \ , &\\
&& \Lambda' = \Lambda - m_2\alpha_2, \quad m_2 \in I\!N \ , \qquad (3.19b)\\
(m_1(\Lambda'), m_2(\Lambda'), m_3(\Lambda'), m_4(\Lambda')) &= (m_1, -m_4, -m_3, -m_2) \ , &\\
&& \Lambda' = \Lambda - m_3\alpha_3, \quad m_3 \in I\!N \ , \qquad (3.19c)\\
(m_1(\Lambda'), m_2(\Lambda'), m_3(\Lambda'), m_4(\Lambda')) &= (-m_3, m_2, -m_1, -m_4) \ , &\\
&& \Lambda' = \Lambda - m_4\alpha_4, \quad m_4 \in I\!N \ . \qquad (3.19d)
\end{aligned}
$$

Since the Verma modules $V^{\Lambda'}$ are isomorphic to submodules of V^Λ they have the same Casimirs. For $\mathcal{C}_2$ this is most obvious from $(3.18b,c)$.

All the above is valid for Verma modules also over $so(5,\mathbb{C})$. Now we shall express conditions (3.12) for the four positive roots taking into account the signatures of our representations (cf. [33]):

$$
\begin{aligned}
m_1 &= \Lambda(H_1) + 1 = 2s_0 + 1 \ , \quad s_0 = \tfrac{1}{2}p \in \tfrac{1}{2}\mathbb{Z}_+ \ , && (3.20a)\\
m_2 &= \Lambda(H_2) + 1 = 1 - E_0 - s_0 \ , \quad E_0 = -\tfrac{1}{2}c \ , && (3.20b)\\
m_3 &= \Lambda(H_3) + 3 = m_1 + 2m_2 \ = \ 3 - 2E_0 \ , && (3.20c)\\
m_4 &= \Lambda(H_4) + 2 = m_1 + m_2 = 2 - E_0 + s_0 \ , && (3.20d)
\end{aligned}
$$

where we have introduced also the traditionally used energy E_0 and spin s_0. For future reference we record an explicit expression for Λ :

$$
\Lambda \ = \ \tfrac{1}{2}(m_1 - 1)\,\alpha_1 \ + \ \tfrac{1}{2}(m_3 - 3)\,\alpha_3 \ = \ s_0\alpha_1 \ - \ E_0\alpha_3 \ . \qquad (3.21)
$$

The eigenvalue of H_3 is called the energy since upon contraction of $so(3,2)$ to the Poincaré algebra, the generator H_3 goes to the translation operator P_0. Analogously, H_1 is the third component of the angular momentum. In terms of E_0, s_0 the Casimir has the expression:

$$
\mathcal{C}_2 \ = \ E_0(E_0 - 3) + s_0(s_0 + 1) + 5/2 \qquad (3.22)
$$

which (up to the additive constant) was the preferred usage in [25],[26],[27]. (In the latter papers the irreducible representations of $so(3,2)$ are denoted by $D(E_0, s_0)$.)

Next we note that if $m_1, m_2 \in I\!N$, then the irreducible representations with non-positive energy $E_0 \ = \ (3 - m_1 - 2m_2)/2$, and spin $s_0 \ = \ (m_1 - 1)/2$, are the non-unitary finite-dimensional representations of $\mathcal{G}$. Their dimension is: $m_1 m_2 m_3 m_4/6$. (For example $m_1\, m_2 = 1$ gives the trivial 1–dimensional representation; the fundamental representations are obtained for $m_1 = 1,\ m_2 = 2$ and $m_1 = 2,\ m_2 = 1$.)

But we are interested in the positive energy UIRs of $\mathcal{G}$ given as follows [20--22] (with $s_0 \in \frac{1}{2}\mathbb{Z}_+$):

$$
\begin{aligned}
&\text{Rac} \ : \ D(E_0, s_0) \ = \ D(1/2, 0) \ , \qquad \text{Di} \ : \ D(E_0, s_0) \ = \ D(1, 1/2) \ , && (3.23a)\\
&D(E_0 > 1/2,\ s_0 \ = \ 0) \ , \quad D(E_0 > 1,\ s_0 \ = \ 1/2) \ , \quad D(E_0 \geq s_0 + 1,\ s_0 \ \geq \ 1). && (3.23b)
\end{aligned}
$$

The first two are the singleton representations discovered by Dirac [20] and the last ones for $E_0 = s_0 + 1$ correspond to the spin-s_0 massless representations. Comparing the list (3.20) with (3.23) we note that (3.20a) holds in all cases of (3.23), while (3.20b) never holds because $m_2 \leq 1/2$. Next, we note that m_3 is a positive integer only for $E_0 = 1/2, 1$, in which case $m_3 = 2, 1$, respectively. Similarly, m_4 is a positive integer only for $E_0 - s_0 = 1$, and that integer is $m_4 = 1$.

The singular vectors corresponding to these cases are (choosing the normalization constants appropriately):

$$
\begin{aligned}
v^{\alpha_1,m_1} &= (X_1^-)^{m_1}\, v_0 \ , \quad m_1 = 2s_0 + 1 \in I\!N \ , && (3.24a)\\
v^{\alpha_3,1} &= (s_0\, X_3^- + X_2^- X_1^-)\, v_0 \ , \quad m_3 = 1 \ , && (3.24b)\\
v^{\alpha_3,2} &= \{(2s_0-1)(2s_0+3)\, X_2^-\, (X_1^-)^2\, X_2^- \ - \ \tfrac{1}{2}(2s_0+1)(2s_0+3)\, (X_2^-)^2\, (X_1^-)^2 \ - \\
&\quad - \ \tfrac{1}{2}(2s_0-1)(2s_0+1)\, (X_1^-)^2\, (X_2^-)^2\} v_0 \ = \\
&= \{\, (1-4s_0^2)\, (X_3^-)^2 \ - \ 4(1-2s_0)\, X_4^-\, X_2^- \ + \\
&\quad + \ 4(1-2s_0)\, X_3^-\, X_2^-\, X_1^- \ - \ 4(X_2^-)^2\, (X_1^-)^2 \ \}\ v_0 \ , \qquad m_3 = 2 \ , && (3.24c)\\
v^{a_4,1} &= \{\, 2s_0(2s_0-1)\, X_4^- + (2s_0-1)\, X_3^-\, X_1^- + X_2^-\, (X_1^-)^2 \ \}\ v_0 \ , \qquad m_4 = 1 \ , && (3.24d)
\end{aligned}
$$

where we have introduced also the non-simple root vectors (choosing the appropriate ordering). Note that (3.24b) for $s_0 = 0$, (3.24c) for $s_0 = 1/2$, and (3.24d) for $s_0 = 0, 1/2$ are composite singular vectors being descendants of (3.24a). (The latter follow also from the general formulae (3.16): for (3.24b, c) from (3.16c) with $m_2 = \frac{1}{2}(m_3 - m_1) = 0$, and for (3.24$d$) from (3.16$d$) with $m_1 = 1, 2 \in \{0, 1, 2 = 2m_4\}$.)

3.3. Invariant differential operators. The main ingredient of the procedure of [29] is that to every singular vector there corresponds an invariant differential operator. Namely, to the singular vector $v_s = v^{\alpha,m_\alpha}$ (cf. (3.14)) of the Verma module V^Λ there corresponds an invariant differential operator

$$
D^{\alpha,m_\alpha} \ : \ C^\Lambda \ \longrightarrow \ C^{\Lambda - m_\alpha \alpha} \qquad (3.25)
$$

given explicitly by:

$$
D^{\alpha,m_\alpha} \ = \ \mathcal{P}^{\alpha,m}(\hat{\mathcal{G}}_-) \qquad (3.26)
$$

where $\mathcal{P}^{\alpha,m}$ is the same polynomial as in (3.14), and $\hat{\mathcal{G}}_-$ symbolizes the right action (3.3).

These operators give rise to the $\mathcal{G}$-invariant equations:

$$
\begin{aligned}
D^{\alpha,m_\alpha}\, \hat{\varphi} \ &= \ \hat{\varphi}' \ , \quad \hat{\varphi} \in C^\Lambda, \quad \hat{\varphi}' \in C^{\Lambda - m_\alpha \alpha}, \quad \alpha \text{ non} - \text{compact} && (3.27a)\\
D^{\alpha,m_\alpha}\, \hat{\varphi} \ &= \ 0 \ , \quad \hat{\varphi} \in C^\Lambda, \quad \alpha \text{ compact} && (3.27b)
\end{aligned}
$$

We recall that a compact root is defined by the property that it has zero value on the dilatation subalgebra $\mathcal{A}$, i.e., in our case, on the generator H_3, which means that only the root α_1 is compact. In fact, equation (3.27b) just expresses the fact that we have induction

from finite-dimensional irreps of the Lorentz group M. In our case, the counterpart of (3.27b) is: $\partial_z^m\,\hat\varphi \;=\; 0$, which is trivially satisfied since all our functions are polynomials of degree $2s_0 = m-1 = p$.

The representations in (3.27a) have the same Casimir. The operators D^{α,m_α} have the intertwining property:

$$D^{\alpha,m_\alpha}\circ\pi_\Lambda(X)\,\hat\varphi \;=\; \pi_{\Lambda'}(X)\circ D^{\alpha,m_\alpha}\,\hat\varphi\;, \qquad \Lambda' = \Lambda - m_\alpha\alpha,\ \ \forall X\in\mathcal{G},\ \ \forall\hat\varphi\in C^\Lambda \tag{3.28}$$

In spite of (3.28) the ERs C^Λ and $C^{\Lambda'}$ are not equivalent but only *partially* equivalent since the differential operators D^{α,m_α} have nontrivial kernels and/or images (for more detailed explanations of these notions we refer to [43]).

For future use we mention one important case in which arises the d'Alembert operator. Namely, we consider $m_1 = 1,\ m_3\in 2I\!\!N$, the singular vectors are (3.16a) and (3.17) (the latter equivalent to (3.16c)). Thus, the relevant equations are:

$$\partial_z\,\hat\varphi \;=0\;, \qquad m_1 = 1\;, \tag{3.29a}$$

$$\begin{aligned} D^{\alpha_3,m_3}\,\hat\varphi \;&= \{(\partial_x - 2z\partial_u)^2 \;-\; 4\,\partial_u\,(\partial_v - z\partial_x + z^2\partial_u)\}^{\frac12 m_3}\,\hat\varphi \;+\; P(\hat{\mathcal{G}}_-)\,\partial_z\,\hat\varphi \;= \\ &= \left(\partial_x^2 \;-\; 4\partial_u\partial_v\right)^{\frac12 m_3}\;\hat\varphi \;=\; \hat\varphi'\;, \qquad m_1 = 1,\ m_3\in 2I\!\!N\,. \end{aligned} \tag{3.29b}$$

Note that the operator $\partial_x^2 \;-\; 4\partial_v\partial_u$ is the d'Alembert operator in M^3 if we identify the coordinates as follows:

$$x \;=\; y_0\;, \quad u \;=\; y_1 - iy_2\;, \quad v \;=\; y_1 + iy_2 \tag{3.30}$$

Then indeed we have:

$$\partial_y^2 \;-\; 4\partial_v\partial_u \;=\; \partial_0^2 - \partial_1^2 - \partial_2^2 \;\equiv\; \Box\;. \tag{3.31}$$

Analogously, the Minkowski length is given by:

$$x^2 - uv \;=\; y_0^2 - y_1^2 - y_2^2 \;\equiv\; y^2\;. \tag{3.32}$$

3.4. Quartet of reducible representations. Here we treat a quartet of reducible representations which is very important since it contains all finite-dimensional irreps of $so(3,2)$. To be more explicit we start with C^Λ such that $m_j\equiv m_j(\Lambda)$ are natural numbers. Then we have the following diagram:

$$\begin{array}{ccc} C^\Lambda & \underset{3,m_3}{\longrightarrow} & C^{\Lambda - m_3\alpha_3} \\ {}^{\ 2}_{m_2}\downarrow & & {}^{\ 4}_{m_2}\uparrow \\ C^{\Lambda - m_2\alpha_2} & \underset{3,m_1}{\longrightarrow} & C^{\Lambda - m_2\alpha_2 - m_1\alpha_3} \end{array} \tag{3.33a}$$

where the numbers at the arrows denote the corresponding differential operators D^{α_k,m'_k}, $k=2,3,4$. It is also very instructive to write the same diagram with explicit signatures:

$$
\begin{array}{ccc}
(m_1,m_2,m_3,m_4) & \underset{3,m_3}{\longrightarrow} & (m_1,-m_4,-m_3,-m_2) \\
{}^{2}_{m_2}\downarrow & & {}^{4}_{m_2}\uparrow \\
(m_3,-m_2,m_1,m_4) & \underset{3,m_1}{\longrightarrow} & (m_3,-m_4,-m_1,m_2)
\end{array}
\tag{3.33b}
$$

The representations in (3.33) have the same Casimir and are partially equivalent. It is natural to ask whether (3.33) is a commutative diagram. Formally, it is, since as differential operators we have:

$$D^{\alpha_3,m_3} \ = \ D^{\alpha_4,m_2} \circ D^{\alpha_3,m_1} \circ D^{\alpha_2,m_2} \tag{3.34}$$

However, the kernels and images of these operators may not coincide,[3] since the operator D^{α_3,m_3} factorizes also according to the second line of (3.16c'):

$$D^{\alpha_3,m_3} \ = \ D^{\alpha_2,m_4} \circ D^{\alpha_1,m_3} \circ D^{\alpha_2,m_2} \tag{3.35}$$

(This second decomposition of D^{α_3,m_3} does not appear on (3.33) since it involves one intermediate space which is unphysical - the z-dependence is not polynomial.)

The finite-dimensional irreps of $so(3,2)$ shall be denoted by E_{m_1,m_2}. They correspond to the holomorphic (or anti-holomorphic) irreps of $so(5,\mathbb{C})$ and have the same dimension: $\dim_{m_1,m_2} \ = \ m_1m_2m_3m_4/6$. In our setting all of these are subspaces of C^Λ in the top-left corner of diagram (3.33) and are obtained in the following way:

$$
\begin{aligned}
E_{m_1,m_2} \ = \ \{ \ f \in C^\Lambda \ : \ m_k = m_k(\Lambda) \ \in \mathbb{N} \ , \quad & D^{\alpha_1,m_1} f \ = \ \partial_z^{m_1} f \ = \ 0, \\
D^{\alpha_2,m_2} \ f \ = \ 0 \ \} &
\end{aligned}
\tag{3.36}
$$

Of course, the first equation is fulfilled for C^Λ by definition, but we include it for completeness. The finite-dimensional subspaces are also in the kernel of the operator D^{α_3,m_3} due to decomposition (3.35).

The diagram (3.33) is important also because it contains effectively an additional integral operator which we shall not consider in detail but just mention at this point. This operator is related to the so-called restricted root system of $\mathcal{G}$ w.r.t. $\mathcal{A}$, where the root spaces are $\mathcal{N}_\pm$. In our case, cf. (2.16), this restricted root system has just one root which coincides with α_3 restricted to $\mathcal{A}$ due to the facts: $[H_3, X_1^\pm] \ = \ 0$, $[H_3, X_k^\pm] \ = \ \pm 2\, X_k^\pm$, $k=2,3,4$. The action of this root on the signatures naturally coincides with the action of α_3, but the action itself is valid for arbitrary signatures. The operator realizing such an action is an integral operator involving the conformal two-point function G_Λ, $\Lambda \cong (m_1,m_2,m_3,m_4)$. For scalar functions ($m_1=1$) the latter is given by:

$$G_\Lambda(y) \ = \ \frac{N(\Lambda)}{(y^2)^{\frac{1}{2}(3+m_3)}} \ , \qquad y^2 = y_0^2 - y_1^2 - y_2^2 \ , \tag{3.37}$$

[3] This question will be discussed in more details in [44].

where $N(\Lambda)$ is a normalization constant related to the Plancherel measure,[4] and the integral operator itself is given by:

$$\begin{aligned} & A_\Lambda \ : \ C^\Lambda \ \longrightarrow \ C^{\Lambda'} \ , \qquad \Lambda' = \Lambda - m_3\alpha_3 \ , \\ & (A_\Lambda\, \hat\varphi)(y) \ = \ \int G_\Lambda(y-y')\ \hat\varphi(y')\ d^3y' \ , \qquad \hat\varphi \in C^\Lambda \ . \end{aligned} \tag{3.38}$$

If $m_3 \notin \mathbb{Z}$ the operators A_Λ and $A_{\Lambda'}$ are inverse to each other:

$$A_{\Lambda'} \circ A_\Lambda \ = \ \mathrm{id}_{C^\Lambda} \ , \qquad A_\Lambda \circ A_{\Lambda'} \ = \ \mathrm{id}_{C^{\Lambda'}} \ . \tag{3.39}$$

For $m_3 \in 2\mathbb{N}$ the operator A_Λ reduces to D^{α_3,m_3} in the same situation (cf. (3.29), (3.30), (3.31)) due to the following:

$$\begin{aligned} & \lim_{\epsilon\mapsto+0} G_{\Lambda(m_3+\epsilon)}(y) \ = \ \lim_{\epsilon\mapsto+0} \frac{N(\Lambda(m_3+\epsilon))}{(y^2)^{\frac12(3+m_3+\epsilon)}} \ \sim \ \lim_{\epsilon\mapsto+0} \frac{\epsilon}{(y^2)^{\frac12(3+m_3+\epsilon)}} \ \sim \\ & \qquad\qquad \sim \ \Box^{\frac12 m_3}\, \delta^3(y) \end{aligned} \tag{3.40}$$

(cf. [48],[43],[49]) and then we have:

$$(A_\Lambda\, \hat\varphi)(y) \ \sim \ \int \Box^{\frac12 m_3}\, \delta^3(y-y')\ \hat\varphi(y')\ d^3y' \ = \ \Box^{\frac12 m_3}\, \hat\varphi(y) \ = \ D^{\alpha_3,m_3}\, \hat\varphi(y) \ . \tag{3.41}$$

For the same values $m_3 \in 2\mathbb{N}$ the operator $A_{\Lambda'}$ remains integral and acts in the *opposite* direction w.r.t. D^{α_3,m_3} on diagram (3.25) (in the case $\alpha = \alpha_3$) and diagram (3.33) (two occurrences). This is the reason we mention the integral operators. (Note that (3.40) remains valid also for $m_3 = 0$ when $\Lambda' = \Lambda$ and the operator A_Λ reduces to the identity operator id_{C^Λ} .)

4. Invariant differential operators and equations related to UIRs

The differential operators related to the unitary cases correspond to the singular vectors in (3.24) and are given as follows:

$$D^{\alpha_1,m} \ = \ \partial_z^m \ , \qquad m = m_1 = p+1 = 2s_0+1 \in \mathbb{N} \ , \tag{4.1a}$$

$$D^{\alpha_3,1} \ = \ s_0\,(\partial_x - 2z\partial_u) \ + \ (\partial_v - z\partial_x + z^2\partial_u)\,\partial_z \ , \qquad m_3 = 1 \ , \tag{4.1b}$$

$$\begin{aligned} D^{\alpha_3,2} \ =& \ (1-4s_0^2)\,(\partial_x - 2z\partial_u)^2 \ - \ 4(1-2s_0)\,\partial_u\,(\partial_v - z\partial_x + z^2\partial_u) \ + \\ & + \ 4(1-2s_0)\,(\partial_x - 2z\partial_u)\,(\partial_v - z\partial_x + z^2\partial_u)\,\partial_z \ - \\ & - \ 4(\partial_v - z\partial_x + z^2\partial_u)^2\,\partial_z^2 \ , \qquad m_3 = 2 \ , \end{aligned} \tag{4.1c}$$

$$\begin{aligned} D^{a_4,1} \ =& \ 2s_0(2s_0-1)\,\partial_u \ + \ (2s_0-1)\,(\partial_x - 2z\partial_u)\,\partial_z \ + \\ & + \ (\partial_v - z\partial_x + z^2\partial_u)\,\partial_z^2 \ , \qquad m_4 = 1 \ . \end{aligned} \tag{4.1d}$$

Next we analyze the important cases separately.

---

[4] The y-dependence of $G_\Lambda(y)$ was given first in [45] for arbitrary space-time dimension D (replace $3+m$ with $D+m$), the group theory interpretation belongs to mathematical [46] and physical [47] work, for the interpretation of $N(\Lambda)$ and more details we refer to [43].

4.1. Rac. In this case the energy and spin are: $E_0 = \frac{1}{2}$, $s_0 = 0$. The equations are (4.1a, c) (cf. also (3.29)):

$$\begin{aligned}
&\partial_z \hat\varphi \ = \ 0 \ , && (4.2a)\\
&D^{\alpha_3,2}\ \hat\varphi \ = \ \{\ (\partial_x - 2z\partial_u)^2 \ - \ 4\,\partial_u\,(\partial_v - z\partial_x + z^2\partial_u) \ + && (4.2b')\\
&+ \ 4\,(\partial_x - 2z\partial_u)\,(\partial_v - z\partial_x + z^2\partial_u)\,\partial_z \ - \ 4(\partial_v - z\partial_x + z^2\partial_u)^2\,\partial_z^2\ \}\,\hat\varphi \ = && (4.2b'')\\
&= \ \left(\partial_x^2 \ - \ 4\partial_u\partial_v\right)\ \hat\varphi \ = \ \Box\,\hat\varphi \ = \ \hat\varphi' \ . && (4.2b)
\end{aligned}$$

Note that the target space $C^{\Lambda'}$, $\Lambda' = \Lambda - 2\alpha_3$, is again of scalar functions ($s'_0 = 0$) and unitary ($E'_0 = \frac{5}{2}$). Thus, $C^{\Lambda'}$ is reducible, since the image $\left(\partial_x^2 \ - \ 4\partial_u\partial_v\right)\ C^\Lambda$ can not be onto $C^{\Lambda'}$, the functions $\hat\varphi'$ on the RHS belong to a proper subspace of $C^{\Lambda'}$. Note that the Verma module $V^{\Lambda'}$ is reducible only w.r.t. (3.20a) so that $m'_1 = m_1 = 1$, thus, the functions of $C^{\Lambda'}$ obey only one differential equation: $\partial_z\,\hat\varphi' \ = \ 0$.

The Rac case is the lowest case of the subfamily in which the invariant operator is directly related to the d'Alembertian, cf. (3.29.)

4.2. Di. In this case the energy and spin are: $E_0 = 1$, $s_0 = \frac{1}{2}$. The equations are (4.1a, b):

$$\begin{aligned}
&\partial_z^2\,\hat\varphi \ = \ 0 \ , && (4.3a)\\
&D^{\alpha_3,1}\ \hat\varphi \ = \ \{\ \tfrac{1}{2}\,(\partial_x - 2z\partial_u) \ + \ (\partial_v - z\partial_x + z^2\partial_u)\,\partial_z\ \}\,\hat\varphi \ = \ \hat\varphi' \ . && (4.3b)
\end{aligned}$$

The invariant equation (4.3b) has the advantage over the usual writing of the equation for the "Di" field (cf. [24]) since it is a scalar equation encompassing all information. It is also more easy to decompose the equation in components by setting

$$\hat\varphi_{\rm Di} \ = \ \hat\varphi_0 \ + \ z\,\hat\varphi_1 \ , \qquad \partial_z\,\hat\varphi_i = 0 \ , \tag{4.4}$$

and analogously for $\hat\varphi'$ in (4.3b). Substituting the above in (4.3b) and extracting the coefficients of the resulting polynomial in z we have:

$$\begin{aligned}
&\tfrac{1}{2}\,\partial_x\,\hat\varphi_0 \ + \ \partial_v\,\hat\varphi_1 \ = \ \hat\varphi'_0 \ , && (4.5a)\\
&\partial_u\,\hat\varphi_0 \ + \ \tfrac{1}{2}\,\partial_x\,\hat\varphi_1 \ = \ \hat\varphi'_1 \ . && (4.5b)
\end{aligned}$$

The first equation is the coefficient at z^0 of (4.3b), the second is at z^1, while the coefficient at z^2 is identically zero.

Note that the target space $C^{\Lambda'}$ in (4.3b), $\Lambda' = \Lambda - \alpha_3$, is again of two-component functions ($s'_0 = \frac{1}{2}$) and unitary ($E'_0 = 2$). Thus, $C^{\Lambda'}$ is reducible, as in the Rac case, and again the Verma module $V^{\Lambda'}$ is reducible only w.r.t. (3.20a) so that $m'_1 = m_1 = 2$, thus, the functions of $C^{\Lambda'}$ obey only one differential equation: $\partial_z^2\,\hat\varphi' \ = \ 0$.

We now consider the kernel of the differential operator $D^{\alpha_3,1}$, i.e., when the RHS of (4.3) and (4.5) are zero. Acting on (4.5a) by ∂_x and on (4.5b) by $-2\partial_v$ and adding together,

we obtain that $\hat\varphi_0$ fulfils the d'Alembert equation: $(\partial_x^2 - 4\partial_v\partial_u)\hat\varphi_0 = 0$. We obtain the same for $\hat\varphi_1$ acting on (4.5b) by ∂_x and on (4.5a) by $-2\partial_u$ and adding together. Thus, we recover the fact that the 'Di' fulfils the d'Alembert equation:

$$\left(\partial_x^2 \ - \ 4\partial_v\partial_u\right) \ \hat\varphi_{\rm Di} \ = \ 0 \ . \tag{4.6}$$

Of course, the full equations (4.3) (or in components (4.5)) contain more information than (4.6).

4.3. Massless representations. In this case the energy and spin are: $E_0 = s_0 + 1$, $s_0 = 1, \frac{3}{2}, \ldots$ The equations are (4.1a, d):

$$\begin{aligned} &\partial_z^{p+1}\,\hat\varphi \ = \ 0 \ , \qquad p = 2s_0 = 2,3,\ldots \ , &(4.7a)\\ &D^{\alpha_4,1}\ \hat\varphi \ = \\ &= \ \{\, p(p-1)\,\partial_u \ + \ (p-1)(\partial_x - 2z\partial_u)\,\partial_z \ + \ (\partial_v - z\partial_x + z^2\partial_u)\,\partial_z^2 \,\}\,\hat\varphi \ = \ \hat\varphi' &(4.7b)\end{aligned}$$

Unlike the singleton cases the target space $C^{\Lambda'}$ is unitary but not massless: it has $E'_0 = s'_0 + 3\,,\ \ s'_0 = s_0 - 1$, $(p' = p-2)$.

Thus, from the point of view of applications to physics these invariant equations relate a field of spin s_0 to a field of spin $s_0 - 1$. In the simplest case a vector field $s_0 = 1$ is coupled to a scalar field.

It is useful to write out (4.7b) in components using: $\hat\varphi \ = \ \sum_{j=0}^{p} z^j\,\hat\varphi_j$, $\hat\varphi' \ = \ \sum_{j=0}^{p-2} z^j\,\hat\varphi'_j$. The result is $p-1$ equations:

$$\begin{gathered}(p-j)(p-j-1)\,\partial_u\,\hat\varphi_j \ + (j+1)(p-j-1)\,\partial_x\,\hat\varphi_{j+1} \ + \ (j+1)(j+2)\,\partial_v\,\hat\varphi_{j+2} \ = \ \hat\varphi'_j \ , \\ j = 0,1,\ldots,p-2 \ .\end{gathered} \tag{4.8}$$

Further we restrict to the kernel of $D^{\alpha_4,1}$. In the simplest case $p = 2$ there is only one equation:

$$2\partial_u\,\hat\varphi_0 \ + \ \partial_x\,\hat\varphi_1 \ + \ 2\partial_v\,\hat\varphi_2 \ = \ 0 \ , \tag{4.9}$$

which can be rewritten as equation for conserved current with the substitution:

$$\hat\varphi_0 \ = \ J_1 - iJ_2 \ , \quad \hat\varphi_1 \ = \ -J_0 \ , \quad \hat\varphi_2 \ = \ J_1 + iJ_2 \ , \tag{4.10}$$

using also (3.30):

$$\partial_0\,J_0 \ - \ \partial_1\,J_1 \ - \ \partial_2\,J_2 \ = \ 0 \ . \tag{4.11}$$

Similarly, for arbitrary $p = 2,3,\ldots,$ there are $p-1$ independent conserved currents $J^{p,j}$, $j = 0,\ldots,p-2$, with components as follows:

$$\begin{aligned} J_0^{p,j} \ &= \ -\,(j+1)(p-j-1)\,\hat\varphi_{j+1} \ , \\ J_1^{p,j} \ &= \ \tfrac{1}{2}\{\,(j+1)(j+2)\,\hat\varphi_{j+2} \ + \ (p-j)(p-j-1)\,\hat\varphi_j\,\} \ , \\ J_2^{p,j} \ &= \ \tfrac{i}{2}\{\,(j+1)(j+2)\,\hat\varphi_{j+2} \ - \ (p-j)(p-j-1)\,\hat\varphi_j\,\} \ . \end{aligned} \tag{4.12}$$

Finally, we would like to mention that all massless representations appear on diagram (3.33) in the case $m_2 = 1$. We would like to parametrize these diagrams by the spin of the massless irrep, i.e., by $s_0 = 1, \frac{3}{2}, \ldots$. In that case the signature of the ER C^Λ on the top-left corner is: $(2s_0 - 1, 1, 2s_0 + 1, 2s_0)$ and this ER contains a finite-dimensional irrep of dimension: $s_0(4s_0^2 - 1)/3$. The massless UIR is contained in the ER in the bottom-right corner of (3.33), while the massive UIR of spin $s_0 - 1$ is in the ER in the top-right corner.

5. Classification of $so(5,\mathbb{C})$ irreps

In view of possible applications relating character formulae to integrability we need first to classify the irreducible HWM over $so(5,\mathbb{C})$. For this we use two important properties of Verma modules [50]:

- Every Verma module V^Λ contains a unique proper maximal submodule I^Λ.
- Every HWM is isomorphic to a factor-module of the Verma module with the same highest weight. (Universality property).

Thus, among the HWM with highest weight Λ there is a unique irreducible one, denoted by L_Λ, i.e.

$$L_\Lambda = V^\Lambda / I^\Lambda \ . \tag{5.1}$$

Clearly, if the Verma module V^L is *irreducible* then $L_\Lambda = V^\Lambda$. Thus, we need to classify the reducible Verma modules.

It is convenient for such classification to use the notion of a multiplet [31] of highest weight modules. We say that a set $\mathcal{M}$ of highest weight modules forms a *multiplet* if 1) $V \in \mathcal{M} \Rightarrow \mathcal{M} \supset \mathcal{M}_V$, where $\mathcal{M}_V$ is the set of all highest weight modules $V' \neq V$ such that either V' is isomorphic to a submodule of V or V is isomorphic to a submodule of V'; 2) $\mathcal{M}$ does not have proper subsets fulfilling 1). It is convenient to depict a multiplet by a connected oriented graph, the vertices of which correspond to the highest weight modules and the arrows between the vertices correspond to the embeddings between the modules. Most often only the embeddings which are not compositions of other embeddings are depicted, since these contain all the relevant information.

Further we shall restrict the notion of a multiplet to the category of Verma modules. We also need the notion of *types of multiplets.* We say that two multiplets belong to the same type, if they are depicted by the same graph and differ only by the values of some parameter(s).

The classification can be summarized as follows. There are four types of multiplets of reducible Verma modules: type $\mathcal{N}$ with four subtypes: $\mathcal{N}_k$, $k = 1,2,3,4$; type $\mathcal{F}_{m_1,m_2}$, $(m_1, m_2 \in \mathbb{N})$; type $\mathcal{S}_{m_1,m_3}$, $(m_1, m_3 \in \mathbb{N},\ \frac{1}{2}(m_1+m_3) \notin \mathbb{Z})$; type $\mathcal{L}$ with two subtypes: $\mathcal{L}_k$, $k = 1,2$, as given explicitly below.

Multiplets of type $\mathcal{N}$ are given as follows. Fix $k = 1,2,3,4$ to fix a subtype $\mathcal{N}_k$. Then the multiplets of this subtype are parametrized by the natural number m_k and are given as follows:

$$V^{\Lambda_k} \ \longrightarrow \ V^{\Lambda_k - m_k\alpha_k} \ , \quad m_k(\Lambda_k) = m_k \in \mathbb{N} \ , \quad m_j(\Lambda_k) \notin \mathbb{N}, \ \ j \neq k \ , \tag{5.2a}$$

$$L_{\Lambda_k} \ = \ V^{\Lambda_k} / V^{\Lambda_k - m_k\alpha_k} \ . \tag{5.2b}$$

Note that we are using the convention that the arrows point to the embedded modules. The modules $V^{\Lambda_k - m_k\alpha_k}$ are irreducible. [The last statement is trivial for $k = 1,2$ from (3.19). For $k = 3$ we need to check that $m_j(\Lambda_3) \notin -\mathbb{N}$ for $j = 2,4$ which is easy since supposing that $m_2(\Lambda_3) \in -\mathbb{N}$ leads to $m_4(\Lambda_3) \in \mathbb{N}$ which is already excluded,

and similarly interchanging $2 \leftrightarrow 4$. For $k = 4$ we need to check that $m_j(\Lambda_4) \notin -I\!N$ for $j = 1, 3$ which is proved similarly to the case $k = 3$.]

Remark: Under the same conditions the ER counterpart of (5.2) is given by (3.25). $\diamondsuit$

Multiplets of type $\mathcal{F}_{m_1,m_2}$ are parametrized by two natural numbers m_1, m_2. They are given as follows:

$$\begin{tikzcd}
& \Lambda_{m_1,m_2} \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\
\Lambda^1_{m_1,m_2} \arrow[d, "s_2"'] \arrow[drr, "s_4"] & & \Lambda^2_{m_1,m_2} \arrow[d, "s_1"] \arrow[dll, "s_3"'] \\
\Lambda^{12}_{m_1,m_2} \arrow[d, "s_1"'] \arrow[drr, "s_4"] & & \Lambda^{21}_{m_1,m_2} \arrow[d, "s_2"] \arrow[dll, "s_3"'] \\
\Lambda^{121}_{m_1,m_2} \arrow[dr, "s_2"'] & & \Lambda^{212}_{m_1,m_2} \arrow[dl, "s_1"] \\
& \Lambda' &
\end{tikzcd}$$

Fig. 1A

$$\begin{tikzcd}
& (m_1, m_2, m_3, m_4) \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\
(-m_1, m_4, m_3, m_2) \arrow[d, "s_2"'] \arrow[drr, "s_4"] & & (m_3, -m_2, m_1, m_4) \arrow[d, "s_1"] \arrow[dll, "s_3"'] \\
(m_3, -m_4, -m_1, m_2) \arrow[d, "s_1"'] \arrow[drr, "s_4"] & & (-m_3, m_4, m_1, -m_2) \arrow[d, "s_2"] \arrow[dll, "s_3"'] \\
(-m_3, m_2, -m_1, -m_4) \arrow[dr, "s_2"'] & & (m_1, -m_4, -m_3, -m_2) \arrow[dl, "s_1"] \\
& (-m_1, -m_2, -m_3, -m_4) &
\end{tikzcd}$$

Fig. 1B

where we have given the multiplets in two ways: on Fig. 1A the Verma modules are depicted by their highest weights, while on Fig. 1B they are given by the explicit signatures. Thus, we see that the parametrizing numbers m_1, m_2 are related to the Verma module $V^{\Lambda_{m_1,m_2}}$ on the top: $m_k = m_k(\Lambda_{m_1,m_2})$. The numbers at the embedding arrows indicate w.r.t. which root is the embedding. All Verma modules of these multiplets, except $V^{\Lambda'}$, are reducible and their weights are given explicitly as follows:

$$\begin{aligned}
&\Lambda_{m_1,m_2}\ , \quad m_k = m_k(\Lambda_{m_1,m_2}) \in I\!N, \quad \forall k\ , \\
&\Lambda^i_{m_1,m_2} = \Lambda_{m_1,m_2} - m_i\alpha_i\ , \quad i = 1, 2\ , \\
&\Lambda^{ij}_{m_1,m_2} = \Lambda_{m_1,m_2} - m_i\alpha_i - m_{j+2}\alpha_j\ , \quad (i,j) = (1,2), (2,1)\ , \\
&\Lambda^{iji}_{m_1,m_2} = \Lambda_{m_1,m_2} - (m_i + m_{i+2})\alpha_i - m_{j+2}\alpha_j\ , \quad (i,j) = (1,2), (2,1)
\end{aligned} \tag{5.3}$$

The weights of the irreducible modules $V^{\Lambda'}$ are: $\Lambda' = \Lambda_{m_1,m_2} - (m_1 + m_3)\alpha_1 - (m_2 + m_4)\alpha_2 = \Lambda - 2m_4\alpha_1 - m_3\alpha_2$. Note that only embeddings which are not compositions of other embeddings are given on Fig. 1. In fact, several composite embeddings according to the second lines of $(3.16c')$ and $(3.16d')$ are present on Fig. 1.

Remark: To obtain the ER counterpart of Fig. 1 we need to exclude four spaces with unphysical signatures ($m'_1 \notin I\!N$), and to restore some operators which were compositions on Fig. 1. Thus, the ER counterpart is given by (3.33). $\diamondsuit$

Multiplets of type $\mathcal{S}_{m_1,m_3}$ are parametrized by two natural numbers m_1, m_3 of different parity so that $\frac{1}{2}(m_1+m_3) \notin Z\!\!\!Z$. They are given in Fig. 2 where as above we have given the multiplet in two ways, and again the parametrizing numbers m_1, m_3 are related to the Verma module V^{Λ^s} on the top: $m_k = m_k(\Lambda^s)$, $k = 1,3$. Note that the supplementary condition on these numbers ensure that $m_k = m_k(\Lambda^s) \notin I\!N$, $k = 2,4$, since $m_2(\Lambda^s) = \frac{1}{2}(m_3 - m_1)$, $m_4(\Lambda^s) = \frac{1}{2}(m_3 + m_1)$. The Verma modules of these multiplets, except $V^{\Lambda''}$, are reducible and their weights are given explicitly as follows:

$$\begin{aligned} &\Lambda^s \ , \quad m_k = m_k(\Lambda^s) \in I\!N, \quad k = 1,3, \quad m_k = m_k(\Lambda^s) \notin I\!N, \quad k = 2,4 \ , \\ &\Lambda^s_k = \Lambda^s - m_k \alpha_k \ , \quad k = 1,3 \ . \end{aligned} \tag{5.4}$$

The weights of the irreducible modules $V^{\Lambda''}$ are: $\Lambda'' = \Lambda^s - m_1\alpha_1 - m_3\alpha_3$.

$$\begin{tikzcd} & \Lambda^s \arrow[dl, "s_1"'] \arrow[dr, "s_3"] & \\ \Lambda^s_1 \arrow[dr, "s_3"'] & & \Lambda^s_3 \arrow[dl, "s_1"] \\ & \Lambda'' & \end{tikzcd}$$

Fig. 2A

$$\begin{tikzcd} & (m_1, m_2, m_3, m_4) \arrow[dl, "s_1"'] \arrow[dr, "s_3"] & \\ (-m_1, m_4, m_3, m_2) \arrow[dr, "s_3"'] & & (m_1, -m_4, -m_3, -m_2) \arrow[dl, "s_1"] \\ & (-m_1, -m_2, -m_3, -m_4) & \end{tikzcd}$$

Fig. 2B

Remark: To obtain the ER counterpart of Fig. 2 we need to exclude two spaces with unphysical signatures ($m'_1 \notin I\!N$). Thus, the ER counterpart is given by (3.25) for $\alpha = \alpha_3$. $\diamondsuit$

Multiplets of type $\mathcal{L}$ are given as follows. Fix $k = 1,2$ to fix a subtype $\mathcal{L}_k$. Then the multiplets are parametrized by the natural number m and are given as follows.

The multiplets of subtype $\mathcal{L}_1$ are given on Fig.3. The weights of the reducible Verma modules are given as follows:

$$\begin{aligned} &\Lambda^1_m \ , \quad m_2(\Lambda^1_m) = m_4(\Lambda^1_m) = m \in I\!N, \quad m_3(\Lambda^1_m) = 2m, \quad m_1(\Lambda^1_m) = 0 \ , \\ &\Lambda^{12}_m = \Lambda^1_m - m\alpha_2 \ , \\ &\Lambda^{121}_m = \Lambda^1_m - m\alpha_4 \ . \end{aligned} \tag{5.5}$$

The weights of the irreducible modules $V^{\Lambda'_m}$ are: $\Lambda'_m = \Lambda^1_m - 2m\alpha_3$. Note that this embedding picture is actually a special case of the factorization of the singular vector v^{α_3, m_3} according to the second line of ($3.16c'$) for $m_1 = 0$, $m_2 = m$.

$$\begin{tikzcd} \Lambda^1_m \arrow[d, "s_2"'] \\ \Lambda^{12}_m \arrow[d, "s_1"'] \\ \Lambda^{121}_m \arrow[d, "s_2"'] \\ \Lambda'_m \end{tikzcd}$$

Fig. 3A

$$\begin{tikzcd} (0,m,2m,m) \arrow[d, "s_2"'] \\ (2m,-m,0,m) \arrow[d, "s_1"'] \\ (-2m,m,0,-m) \arrow[d, "s_2"'] \\ (0,-m,-2m,-m) \end{tikzcd}$$

Fig. 3B

The multiplets of subtype $\mathcal{L}_2$ are given on Fig.4. The weights of the reducible Verma modules are given as follows:

$$\begin{aligned} &\Lambda^2_m \ , \quad m_1(\Lambda^2_m) = m_3(\Lambda^2_m) = m_4(\Lambda^2_m) = m \in I\!\!N, \quad m_2(\Lambda^2_m) = 0 \ , \\ &\Lambda^{21}_m = \Lambda^2_m - m\alpha_1 \ , \\ &\Lambda^{212}_m = \Lambda^2_m - m\alpha_3 \ . \end{aligned} \tag{5.6}$$

The weights of the irreducible modules $V^{\Lambda''_m}$ are: $\Lambda''_m = \Lambda^2_m - m\alpha_4$. Note that this embedding picture is a special case of the factorization of the singular vector v^{α_4,m_4} according to the second line of $(3.16d')$ for $m_2 = 0$, $m_1 = m$.

We can summarize the classification of the reducible Verma modules as follows:
Proposition: All reducible Verma modules over $so(5,\mathbb{C})$ are explicitly parametrized by the following highest weights: Λ_k from (5.2), Λ_{m_1,m_2} , $\Lambda^i_{m_1,m_2}$, $\Lambda^{ij}_{m_1,m_2}$, $\Lambda^{iji}_{m_1,m_2}$ from (5.3), Λ^s, Λ^s_k from (5.4), Λ^1_m , Λ^{12}_m , Λ^{121}_m from (5.5), Λ^2_m , Λ^{21}_m , Λ^{212}_m from (5.6). The same highest weights parametrize all irreps L_Λ for which V^Λ are reducible. All irreps are infinite-dimensional, except $L_{m_1,m_2} \equiv L_{\Lambda_{m_1,m_2}}$. The latter are the holomorphic finite-dimensional irreps of $so(5,\mathbb{C})$ of dimension $m_1m_2m_3m_4/6$, which are isomorphic to the finite-dimensional non-unitary irreps of $so(3,2)$. $\diamondsuit$

We note important property of reduction of multiplets. The multiplet $\mathcal{L}_1$, would be obtained from the multiplets of type $\mathcal{F}_{m_1,m_2}$ if we formally set $m_1 = 0$, (and then $m_2 = m$), since the eight modules coincide pairwise reducing the multiplet to four modules (the notation was chosen suggestively). Analogously, the multiplet $\mathcal{L}_2$, would be obtained from the multiplets of type $\mathcal{F}_{m_1,m_2}$ if we formally set $m_2 = 0$, (and then $m_1 = m$). The opposite process would go from Fig. 2 if we suppose that $m_2, m_4 \in I\!\!N$, since then the two singular vectors v^{α_3,m_3} would decompose according to the second line of $(3.16c')$ and four new reducible modules would appear restoring Fig. 1.

$$\begin{tikzcd}
\Lambda^2_m \arrow[d, "s_1"'] \\
\Lambda^{21}_m \arrow[d, "s_2"'] \\
\Lambda^{212}_m \arrow[d, "s_1"'] \\
\Lambda''_m
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
(m,0,m,m) \arrow[d, "s_1"'] \\
(-m,m,m,0) \arrow[d, "s_2"'] \\
(m,-m,-m,0) \arrow[d, "s_1"'] \\
(-m,0,-m,-m)
\end{tikzcd}$$

Fig. 4A Fig. 4B

6. Character formulae of AdS irreps and possible applications to integrability

6.1. Character formulae of AdS irreps. The question of reducibility of Verma modules is closely related to the corresponding Weyl groups. In particular, whenever (3.12) is fulfilled, then the shifted weight is given by the corresponding Weyl reflection. We recall that by definition the Weyl reflection in $\mathcal{H}^*$ is the following action:

$$s_\alpha(\Lambda) \doteq \Lambda - (\Lambda, \alpha^\vee)\, \alpha \tag{6.1}$$

Using this we can define also a 'dot'-action:

$$s_\alpha \cdot (\Lambda) \doteq s_\beta\, (\Lambda + \rho) - \rho\ . \tag{6.2}$$

Thus, we see that the shifted weight Λ' in (3.13) is given by (using (3.12)):

$$\Lambda' = \Lambda - m\beta = \Lambda - (\Lambda + \rho, \beta^\vee)\, \beta = s_\beta\, (\Lambda + \rho) - \rho = s_\beta \cdot (\Lambda) \tag{6.3}$$

The group generated by the Weyl reflections is called the Weyl group W. In fact, it is generated by the simple reflections s_α, where α runs through all simple roots. Since the Weyl group is generated by the simple reflections then every element $w \in W$ may be written as the product of some simple reflections. Every such product which uses a minimal number of simple reflections is called a reduced expression or reduced form for w. The number of simple reflections in a reduced form is called the length of w and is denoted by $\ell(w)$. In our case, $\mathcal{G}^{\mathbb{C}} = so(5, \mathbb{C}) = B_2$, the Weyl group W_{B_2} has eight elements explicitly given in reduced form by $(s_k \equiv s_{\alpha_k})$:

$$\begin{aligned} W_{B_2} = \{\ & 1,\ s_1\,,\ s_2\,,\ s_1\, s_2\,,\ s_2\, s_1\,,\ s_3 = s_2\, s_1\, s_2\,,\ \ s_4 = s_1\, s_2\, s_1\,, \\ & s_1\, s_2\, s_1\, s_2 = s_2\, s_1\, s_2\, s_1\ \}\ . \end{aligned} \tag{6.4}$$

Let $\mathcal{G}$ be any simple Lie algebra. Let Γ , (resp. Γ_+), be the set of all integral, (resp. integral dominant), elements of $\mathcal{H}^*$, i.e., $\Lambda \in \mathcal{H}^*$ such that $(\Lambda, \alpha_i^\vee) \in \mathbb{Z}$, (resp. $\mathbb{Z}_+$), for all simple roots α_i . We recall that for each invariant subspace $V \subset U(\mathcal{G}_+) \otimes v_0 \cong V^\Lambda$ we have the following decomposition

$$V \ = \ \bigoplus_{\mu \in \Gamma_+} V_\mu \ \ , \quad V_\mu \ = \ \{ u \in V \mid H_k u = (\Lambda + \mu)(H_k) u, \ \ \forall\ H_k \} \ . \tag{6.5}$$

(Note that $V_0 \ = \ \mathbb{C}\, v_0$.) Following [50],[51] let $E(\mathcal{H}^*)$ be the associative abelian algebra consisting of the series $\sum_{\mu \in \mathcal{H}^*} c_\mu \, e^\mu$, where $c_\mu \in \mathbb{C}$, $c_\mu = 0$ for μ outside the union of a finite number of sets of the form $D(\Lambda) = \{\mu \in \mathcal{H}^* | \mu \leq \Lambda\}$, using any ordering of $\mathcal{H}^*$.

The character of V is defined by :

$$ch\ V \ \ = \ \sum_{\mu \in \Gamma_+} (\dim\ V_\mu)\, e^{\Lambda + \mu} \ \ = \ \ e^\Lambda \sum_{\mu \in \Gamma_+} (\dim\ V_\mu)\, e^\mu \ . \tag{6.6}$$

We recall [50] that for $V = V^\Lambda$ we have $\dim V_\mu = P(\mu)$, $P(\mu)$ is a generalized partition function, $P(\mu) = \#$ of ways μ can be presented as a sum of positive roots β_j , each root taken with its multiplicity $m_j = \dim\ \mathcal{G}_{\beta_j}$, (here $m_j = 1$), $P(0) \equiv 1$. Analogously we use [50] to obtain :

$$ch\ V^\Lambda \ \ = \ \ e^\Lambda \sum_{\mu \in \Gamma_+} P(\mu)\, e^\mu \ \ = \ \ e^\Lambda \prod_{\alpha \in \Delta^+} (1 - e^\alpha)^{-1} \ . \tag{6.7}$$

The Weyl character formula for the finite-dimensional irreducible highest weight representations over $\mathcal{G}$ has the form [50]:

$$ch\ L_\Lambda \ \ = \ \ ch\ V^\Lambda \sum_{w \in W} (-1)^{\ell(w)}\, e^{w \cdot \Lambda - \Lambda} \ \ = \ \ \sum_{w \in W} (-1)^{\ell(w)}\ ch\ V^{w \cdot \Lambda} \ . \tag{6.8}$$

Let $\mathcal{G} = so(5, \mathbb{C})$, or $\mathcal{G} = so(3,2)$. Then the character formula for the finite-dimensional irreps (6.8) can be rewritten as

$$\begin{aligned} ch\ L_{m_1,m_2} \ \ &= \ ch\ V^{\Lambda_{m_1,m_2}} \ (\ 1 \ - \ e^{\alpha_1 m_1} \ - \ e^{\alpha_2 m_2} \ + \ e^{\alpha_1 m_1} e^{\alpha_2 (m_1 + m_2)} \ + \\ &+ \ e^{\alpha_1 m_1 + 2 m_2} e^{\alpha_2 m_2} \ - \ e^{\alpha_4 m_4} \ - \ e^{\alpha_3 m_3} \ + \ e^{2\alpha_1 m_4} e^{\alpha_2 m_3} \ ) \ . \end{aligned} \tag{6.9}$$

Derivation of (6.9): It is easy to derive the above formula using the embedding diagram on Fig. 1A. We use the general fact that from (5.1) follows:

$$ch\ L_\Lambda = ch\ V^\Lambda - ch\ I^\Lambda \ . \tag{6.10}$$

Now it is clear that $ch\ I^{\Lambda_{m_1,m_2}}$ contains the terms

$$ch\ V^{\Lambda^1_{m_1,m_2}} + ch\ V^{\Lambda^2_{m_1,m_2}} \tag{6.11}$$

since $V^{\Lambda^1_{m_1,m_2}}$ and $V^{\Lambda^2_{m_1,m_2}}$ are submodules of $V^{\Lambda_{m_1,m_2}}$, however, the sum in (6.11) is larger than $ch\ I^{\Lambda_{m_1,m_2}}$ because of the overlap. Indeed, both $V^{\Lambda^1_{m_1,m_2}}$ and $V^{\Lambda^2_{m_1,m_2}}$ contain as submodules $V^{\Lambda^{12}_{m_1,m_2}}$ and $V^{\Lambda^{21}_{m_1,m_2}}$. Thus the latter are contained in (6.11) two times. To correct this over-counting we should subtract their characters and replace (6.11) by:

$$ch\ V^{\Lambda^1_{m_1,m_2}} + ch\ V^{\Lambda^2_{m_1,m_2}} - ch\ V^{\Lambda^{12}_{m_1,m_2}} - ch\ V^{\Lambda^{21}_{m_1,m_2}} \tag{6.12}$$

However, now there is under-counting since both $V^{\Lambda^{12}_{m_1,m_2}}$ and $V^{\Lambda^{21}_{m_1,m_2}}$ contain as submodules $V^{\Lambda^{121}_{m_1,m_2}}$ and $V^{\Lambda^{212}_{m_1,m_2}}$. To correct this under-counting we should add their characters and replace (6.12) by:

$$ch\ V^{\Lambda^1_{m_1,m_2}} + ch\ V^{\Lambda^2_{m_1,m_2}} - ch\ V^{\Lambda^{12}_{m_1,m_2}} - ch\ V^{\Lambda^{21}_{m_1,m_2}} + ch\ V^{\Lambda^{121}_{m_1,m_2}} + ch\ V^{\Lambda^{212}_{m_1,m_2}} \tag{6.13}$$

Now the module $V^{\Lambda'}$ is over-counted. Thus, the final formula for $ch\ I^{\Lambda_{m_1,m_2}}$ is:

$$\begin{aligned} ch\ I^{\Lambda_{m_1,m_2}} &= ch\ V^{\Lambda^1_{m_1,m_2}} + ch\ V^{\Lambda^2_{m_1,m_2}} - ch\ V^{\Lambda^{12}_{m_1,m_2}} - ch\ V^{\Lambda^{21}_{m_1,m_2}} + \\ &+ ch\ V^{\Lambda^{121}_{m_1,m_2}} + ch\ V^{\Lambda^{212}_{m_1,m_2}} - ch\ V^{\Lambda'} \end{aligned} \tag{6.14}$$

Substituting (6.14) in (6.10), using the explicit values of the various highest weights from (5.3) we recover (6.9). $\diamondsuit$

The character formulae for the infinite-dimensional irreducible highest weight representations involve less terms than in (6.9) since the maximal invariant submodules I^Λ of V^Λ are smaller. It is easy to derive these using the same considerations as above, so we just list the results.

- The character formulae for the irreps with highest weights Λ_1 from (5.2), $\Lambda^{212}_{m_1,m_2}$ from (5.3), Λ^s_3 from (5.4), Λ^{12}_m from (5.5), Λ^2_m and Λ^{212}_m from (5.6), are:

$$\begin{aligned} ch\ L_\Lambda &= \sum_{w\in W_1} (-1)^{\ell(w)}\ ch\ V^{w\cdot\Lambda} = ch\ V^\Lambda\ (1-e^{m_1\alpha_1})\ , \\ W_1 &= \{\ 1,\ s_1\ \} \end{aligned} \tag{6.15}$$

  where Λ denotes all highest weights under consideration and m_1 should be replaced by m for the cases from (5.5) and (5.6).

- The character formula for the irreps with highest weights Λ_2 from (5.2), $\Lambda^{121}_{m_1,m_2}$ from (5.3), Λ^1_m and Λ^{121}_m from (5.5), Λ^{21}_m from (5.6), is:

$$\begin{aligned} ch\ L_\Lambda &= \sum_{w\in W_2} (-1)^{\ell(w)}\ ch\ V^{w\cdot\Lambda} = ch\ V^\Lambda\ (1-e^{m_2\alpha_2})\ , \\ W_2 &= \{\ 1,\ s_2\ \} \end{aligned} \tag{6.16}$$

  where Λ denotes all highest weights under consideration and m_2 should be replaced by m for the cases from (5.5) and (5.6).

- The character formula for the irreps with highest weights Λ_3 from (5.2), Λ_1^s from (5.4), is

$$\begin{aligned} ch\ L_\Lambda \ &= \sum_{w\in W_3} (-1)^{\ell(w)}\ ch\ V^{w\cdot\Lambda} \ = \ ch\ V^\Lambda\ (1-e^{m_3\alpha_3})\ , \\ W_3 \ &= \ \{\ 1,\ s_3\ \} \end{aligned} \tag{6.17}$$

where Λ denotes all highest weights under consideration.

- The character formula for the irreps with highest weights Λ_4 from (5.2) is

$$\begin{aligned} ch\ L_{\Lambda_4} \ &= \sum_{w\in W_4} (-1)^{\ell(w)}\ ch\ V^{w\cdot\Lambda_4} \ = \ ch\ V^{\Lambda_4}\ (1-e^{m_4\alpha_4})\ , \\ W_4 \ &= \ \{\ 1,\ s_4\ \}\ . \end{aligned} \tag{6.18}$$

- The character formula for the irreps with highest weights Λ^s from (5.4) is

$$\begin{aligned} ch\ L_{\Lambda^s} \ &= \sum_{w\in W^s} (-1)^{\ell(w)}\ ch\ V^{w\cdot\Lambda^s} \ = \\ &= ch\ V^{\Lambda^s}\ (\ 1\ -\ e^{\alpha_1 m_1}\ -\ e^{\alpha_3 m_3}\ +\ e^{\alpha_1 m_1+\alpha_3 m_3}\ )\ , \\ W^s \ &= \ \{\ 1,\ s_1\,,\ s_3\,,\ s_1 s_3\ \}\ =\ W_1\ \times\ W_3 \end{aligned} \tag{6.19}$$

- The character formula for the irreps with highest weights $\Lambda^{12}_{m_1,m_2}$ from (5.3) is

$$\begin{aligned} ch\ L_{\Lambda^{12}_{m_1,m_2}} \ &= \sum_{w\in W^{12}} (-1)^{\ell(w)}\ ch\ V^{w\cdot\Lambda^{12}_{m_1,m_2}} \ = \\ &= ch\ V^{\Lambda^{12}_{m_1,m_2}}\ (\ 1\ -\ e^{\alpha_1 m_3}\ -\ e^{\alpha_4 m_2}\ +\ e^{\alpha_1 m_3+\alpha_2 m_2}\ )\ , \\ W^{12} \ &= \ \{\ 1,\ s_1\,,\ s_2 s_1\,,\ s_1 s_2 s_1\ \} \end{aligned} \tag{6.20}$$

- The character formula for the irreps with highest weights $\Lambda^{21}_{m_1,m_2}$ from (5.3) is

$$\begin{aligned} ch\ L_{\Lambda^{21}_{m_1,m_2}} \ &= \sum_{w\in W^{21}} (-1)^{\ell(w)}\ ch\ V^{w\cdot\Lambda^{21}_{m_1,m_2}} \ = \\ &= ch\ V^{\Lambda^{21}_{m_1,m_2}}\ (\ 1\ -\ e^{\alpha_2 m_4}\ -\ e^{\alpha_3 m_1}\ +\ e^{\alpha_1 m_1+\alpha_2 m_4}\ )\ , \\ W^{21} \ &= \ \{\ 1,\ s_2\,,\ s_1 s_2\,,\ s_2 s_1 s_2\ \} \end{aligned} \tag{6.21}$$

- The character formula for the irreps with highest weights $\Lambda^{1}_{m_1,m_2}$ from (5.3) is

$$\begin{aligned} ch\ L_{\Lambda^{1}_{m_1,m_2}} \ &= \sum_{w\in W^{1}} (-1)^{\ell(w)}\ ch\ V^{w\cdot\Lambda^{1}_{m_1,m_2}} \ = \\ &= ch\ V^{\Lambda^{1}_{m_1,m_2}}\ (\ 1\ -\ e^{\alpha_2 m_4}\ -\ e^{\alpha_4 m_2}\ +\ e^{\alpha_1 m_3+\alpha_2 m_4}\ + \\ &\quad +\ e^{\alpha_2 m_4+\alpha_4 m_2}\ -\ e^{\alpha_1 m_3+\alpha_2 m_3}\ )\ , \\ W^{1} \ &= \ \{\ 1,\ s_2\,,\ s_1 s_2\,,\ s_2 s_1 s_2\,,\ s_1 s_2 s_1\,,\ s_2 s_1 s_2 s_1\ \} \end{aligned} \tag{6.22}$$

• The character formula for the irreps with highest weights $\Lambda^2_{m_1,m_2}$ from (5.3) is

$$\begin{aligned} ch\ L_{\Lambda^2_{m_1,m_2}} \ &= \ \sum_{w\in W^2} (-1)^{\ell(w)}\ ch\ V^{w\cdot\Lambda^2_{m_1,m_2}} \ = \\ &= ch\ V^{\Lambda^2_{m_1,m_2}} \ (\ 1\ -\ e^{\alpha_1 m_3}\ -\ e^{\alpha_3 m_1}\ +\ e^{\alpha_1 m_3+\alpha_2 m_4}\ + \\ &\quad +\ e^{\alpha_1 m_3+\alpha_3 m_1}\ -\ e^{2\alpha_1 m_4+\alpha_2 m_4}\ )\ , \\ W^2 \ &= \ \{\ 1,\ s_1\ ,\ s_2 s_1\ ,\ s_2 s_1 s_2\ ,\ s_1 s_2 s_1\ ,\ s_2 s_1 s_2 s_1\ \} \end{aligned} \tag{6.23}$$

Note that each of the Weyl groups W_k, $k = 1,2,3,4$, is isomorphic to the $A_1 = sl(2)$ Weyl group, while the Weyl group W^s is the direct product of two such A_1 Weyl groups. In contrast, the subsets of W over which is carried the summation in the last four cases, namely, W^1, W^2, W^{12}, W^{21} are not considered subgroups of W, since then the elements of these subsets will generate the whole W.

6.2. Character formulae of positive energy UIRs. Here we apply the character formulae of the previous subsection to the positive energy UIRs of $so(3,2)$.

6.2.1. Rac. We have $m_1 = 1$, $m_2 = 1/2$, i.e., we have a special case of (6.19):

$$\begin{aligned} ch\ L_{\rm Rac} \ &= \ ch\ V^{\Lambda^s}\ (\ 1\ -\ e^{\alpha_1}\ -\ e^{2\alpha_3}\ +\ e^{\alpha_1+2\alpha_3}\ )\ = && (6.24a)\\ &= \ e(\Lambda^s)\ (\ 1\ -\ e^{2\alpha_3}\ )/(1-e^{\alpha_2})(1-e^{\alpha_3})(1-e^{\alpha_4})\ = && (6.24b)\\ &= \ e(\Lambda^s)\ (\ 1\ +\ e^{\alpha_3}\ )/(1-e^{\alpha_2})(1-e^{\alpha_4})\ = && (6.24c)\\ &= \ e(\Lambda^s)\ \sum_{n=0}^{\infty} e^{n\alpha_3} \sum_{p=-n}^{n} e^{p\alpha_1}\ = && (6.24d)\\ &= \ e(\Lambda)\ \sum_{n=0}^{\infty} \sum_{p=-n}^{n} e^{(n-|p|)\alpha_3}\ t'^{|p|}\ , && (6.24e) \end{aligned}$$

where

$$t' \ = \ \begin{cases} e(\alpha_4) & \text{for} \quad p\geq 0\ , \\ e(\alpha_2) & \text{for} \quad p<0\ . \end{cases}$$

Character formula (6.24) is equivalent to the spectrum description given in [33] and clearly each term has different weight from all others, which explains the terminology of singleton.

6.2.2. Di. We have $m_1 = 2$, $m_2 = -1/2$, i.e., again a special case of (6.19):

$$\begin{aligned} ch\ L_{\rm Di} \ &= \ ch\ V^{\Lambda^s}\ (\ 1\ -\ e^{2\alpha_1}\ -\ e^{\alpha_3}\ +\ e^{2\alpha_1+\alpha_3}\ )\ = && (6.25a)\\ &= \ e(\Lambda^s)\ (\ 1\ -\ e^{2\alpha_1}\ )/(1-e^{\alpha_1})(1-e^{\alpha_2})(1-e^{\alpha_4})\ = && (6.25b)\\ &= \ e(\Lambda^s)\ (\ 1\ +\ e^{\alpha_1}\ )/(1-e^{\alpha_2})(1-e^{\alpha_4})\ = && (6.25c)\\ &= \ e(\Lambda^s)\ \sum_{n=0}^{\infty} e^{n\alpha_3} \sum_{p=-n}^{n+1} e^{p\alpha_1}\ = && (6.25d)\\ &= \ e(\Lambda^s)\ \sum_{n=0}^{\infty} e^{n\alpha_2} \sum_{r=0}^{2n+1} e^{r\alpha_1}\ )\ . && (6.25e) \end{aligned}$$

Character formula (6.25) is equivalent to the singleton spectrum description given in [33].

6.2.3. Spin zero. Next we consider the case $s_0 = 0$, $E_0 > 1/2$, cf. (3.23), (3.24) and the text in-between. We have only the singular vector $v^{\alpha_1,1}$. This is clear for $E_0 \neq 1$, while for $E_0 = 1$ one should note that for $s_0 = 0$ the singular vectors $v^{\alpha_3,1}$ in (3.24c) and $v^{\alpha_4,1}$ in (3.24d) are descendants of $v^{\alpha_1,1}$. In fact, for $E_0 \neq 1$ the Verma module is V^{Λ_1} from a multiplet of subtype $\mathcal{N}_1$ for parameter $m_1 = 1$, while for $E_0 = 1$ the Verma module is $V^{\Lambda_1^2}$ from a multiplet of subtype $\mathcal{L}_2$ for parameter $m = 1$. Thus, the character formula is (6.15).

6.2.4. Spin 1/2. Analogously for $s_0 = 1/2$, $E_0 > 1$, we have only the singular vector $v^{\alpha_1,2}$. This is clear for $E_0 \neq 3/2$, while for $E_0 = 3/2$ one should note that for $s_0 = 1/2$ the singular vector $v^{\alpha_4,1}$ in (3.24d) is descendant of $v^{\alpha_1,2}$. In fact, for $E_0 \neq 3/2$ the Verma module is V^{Λ_1} from a multiplet of subtype $\mathcal{N}_1$ for parameter $m_1 = 2$, while for $E_0 = 3/2$ the Verma module is $V^{\Lambda_1^{12}}$ from a multiplet of subtype $\mathcal{L}_1$ for parameter $m = 1$. Thus, the character formula is (6.15) as in the previous case.

6.2.5. Higher spins. Analogously for $s_0 \geq 1$, $E_0 > s_0+1$, cf. (3.23b), we have only the singular vector v^{α_1,m_1}, $m_1 = 2s_0+1$, since $m_2, m_3, m_4 \notin I\!N$. Thus, the Verma module is V^{Λ_1} from a multiplet of subtype $\mathcal{N}_1$ for parameter $m_1 \geq 3$, and the character formula is (6.15) as in the previous case.

6.2.6. Massless irreps. Finally we consider the massless representations with $E_0 = s_0 + 1$, $s_0 \geq 1$. We have two singular vectors: v^{α_1,m_1}, $m_1 = 2s_0 + 1$, and $v^{\alpha_4,1}$. The signature is: $(m_1, m_2, m_3, m_4) = (2s_0 + 1, -2s_0, 1 - 2s_0, 1)$. This signature appears as the Verma module $V^{12}_{2s_0-1,1}$ of the multiplet $\mathcal{F}_{2s_0-1,1}$. Thus, the character formula (found first in [33]) is a special case of (6.20):

$$\begin{aligned} ch\ L_{\Lambda^{12}_{m-2,1}} &= ch\ V^{\Lambda^{12}_{m-2,1}}\ (\ 1\ -\ e^{m\alpha_1}\ -\ e^{\alpha_4}\ +\ e^{m\alpha_1+\alpha_2}\ )\ , \\ & m = m_1 = 2s_0 + 1 \end{aligned} \tag{6.26}$$

Thus, the massless UIRs are in one-parameter family of multiplets $\mathcal{F}_{2s_0-1,1}$ where in the Verma modules $V^{\Lambda_{2s_0-1,1}}$ on the top of the multiplets are found the finite-dimensional irreps of dimension: $s_0(4s_0^2 - 1)/3$, $s_0 = 1, \frac{3}{2}, \ldots$ For the lowest possible value $s_0 = 1$ the finite-dimensional irrep is the trivial one-dimensional irrep of $so(5,I\!\!\!C)$ (and of $so(3,2)$).

6.3. Possible applications to integrability. The classical and quantum integrability/solvability of Calogero-Moser-Sutherland systems [52,53,54,55,56] has attracted a lot of attention for the last 30 years. A very important contribution for the group-theoretic understanding was made by Olshanetsky and Perelomov who extended the original models from the A root system to the B, C and D root systems [57],[58]. For a recent overview we refer to [59] and references therein.

For our setting the interesting aspect is the relation between Coxeter groups and integrability of these systems. More specifically, we are interested in the relation of integrable systems to characters of irreps of simple Lie algebras, cf. [60] and references therein. To illustrate the possible applications of the results of this paper to integrability we first look at the case of Dirac singletons.

In both singleton cases the character formula is given in (6.19) and involves summation over the group W^s which is the direct product of two A_1 Weyl groups. The group W^s can be characterized as generated by the short root reflections s_1 and s_3. Actually, this is a manifestation of a general phenomena since the sets of short and long roots are invariant under the action of the whole Weyl group. Thus, we may also write:

$$W_{B_2}^{\rm short} = W_{A_1} \times W_{A_1} . \tag{6.27}$$

Thus, irreps for which holds (6.19), in particular, the two Dirac singletons, would be related to integrable systems described by the orbit of short roots of the B_2 root system [57],[58],[59].

The irreps with character formulae (6.15),(6.16),(6.17),(6.18) would be related to integrable systems described by the A_1 root system.

All the above could also related to different Toda-like models (for a recent review, cf. [61]) through their character formulae similarly to the application of Virasoro characters in [62], which should be extendable to the supersymmetry setting [63].

More interesting would be models using character formulae (6.20),(6.21),(6.22),(6.23) since in these cases summation is over subsets of W_{B_2} which can not be considered subgroups of W_{B_2}. Thus, these would be new models!

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